The normalized trace on a finite CAR stage

`MathlibAnnex.CStarAlgebra.CAR.stageTrace`

def

The normalized matrix trace provides a positive tracial functional compatible with the CAR inclusions.

Statement

For $n \in \mathbb{N}$, let $A_n = M_{2^n}(\mathbb{C})$ with its operator norm. The finite-stage trace is the continuous complex-linear functional $\tau_n : A_n \to \mathbb{C}$ whose value is $\tau_n(a) = 2^{-n} \sum_{i=0}^{2^n-1} a_{ii}$. The factor 2^{-n} makes $\tau_n(I) = 1$.

Definition

Take the normalized diagonal sum as a linear map. Each diagonal entry satisfies $|a_{ii}| \leq \|a\|$, so the triangle inequality gives $|2^{-n} \sum_i a_{ii}| \leq 2^{-n} 2^n \|a\| = \|a\|$. This bound supplies continuity. Positivity follows by expanding the diagonal of a^*a into sums of squared absolute values; the trace identity follows by interchanging the two finite summations. Amplification repeats every diagonal entry twice, exactly cancelling the change in normalization.

Assumptions

The index n may be zero, in which case $A_0 = \mathbb{C}$. No representation, choice of vector, or limiting operation enters this definition.

Conclusion

The functional is bounded by $|\tau_n(a)| \leq \|a\|$, is positive on a^*a, and satisfies $\tau_n(ab) = \tau_n(ba)$. Under the standard CAR inclusion $s_n : A_n \to A_{n+1}$, given by $a \otimes I_2$ and coordinate reindexing, $\tau_{n+1}(s_n(a)) = \tau_n(a)$. These are the separately proved properties cited below.

Main citations

- Definition and its exact construction · Exact source
- Normalized diagonal sum · Exact source
- Entrywise estimate giving continuity · Exact source
- Bound for the continuous trace · Exact source
- Normalization at the identity · Exact source
- Positivity on squares · Exact source
- Cyclicity of the finite trace · Exact source
- Compatibility with amplification · Exact source
- Trace of the distinguished projection · Exact source

Lean source signature (exact)

```
noncomputable def stageTrace (n : ℕ) : Stage n →L[ℂ] ℂ :=
  (stageTraceLinear n).mkContinuous 1 fun a ↦ by
    simpa only [one_mul] using norm_stageTraceLinear_le n a
```

> Here `Stage n` is A_n, `stageTraceLinear n` is the normalized diagonal sum, and `mkContinuous 1` equips it with the bound 1. The displayed right-hand side is the full definition of `stageTrace`. The formula for its underlying linear map and the proofs of its properties are separate exact citations.

Lean realization notes

The root projection E_{00} has trace 2^{-n}; the root-coordinate functional $a \mapsto a_{00}$ instead takes value 1 there. Thus the trace and the product-vector functional serve different purposes. The declaration here defines a finite-stage functional, not a new assertion about a trace on the completion.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

Card UID: lfh:lfh-declaration-card:sha256:0047cafd9294f990a20362b56c9f7b905407ae2c02f2b9df80a3dba01bf1254b

Card revision: 1 · SHA-256: 45e6017f58cbdedcc1be3088d3b58abd517ca8d0efe97b5f8680a335352d7ccd

Exposition revision: 2 · SHA-256: 0383e61e8449db7d28fa9398e2ae0163b6b0a2d391263673f75789b3ad736e58

Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73

Header SHA-256: 2343c37755c623f18598332050ca73a4628ed407fb235b9322a900929e20e2c1