Purifying a vector state on a finite CAR stage

`MathlibAnnex.CStarAlgebra.CAR.exists_unitVector_vectorFunctional_eq_on_stage`

theorem

Every finite-stage restriction of a unit vector state can be realized by a unit vector in any irreducible CAR representation.

Statement

Let A be the completed CAR algebra: the norm completion of the increasing union of $A_n = M_{2^n}(\mathbb{C})$ under the unital embeddings $a \mapsto a \otimes I_2$ (with the fixed coordinate reindexing). Write $\iota_n : A_n \to A$ for the canonical isometric inclusion and $e_{ij}^{(n)} = \iota_n(E_{ij})$ for its matrix units. A representation is a unital complex star homomorphism into the bounded operators on a complex Hilbert space. Irreducibility means nonzero action and no proper nonzero closed reducing subspace. We use the inner product conjugate linear in its first argument and write $\varphi_{\pi,v}(a) = \langle v, \pi(a)v\rangle$ for the vector functional. Fix an irreducible representation $\pi : A \to B(H)$, any representation $\sigma : A \to B(K)$, and a unit vector $\eta \in K$. For every stage n there is a unit vector $\xi \in H$ such that $\varphi_{\pi,\xi}(\iota_n(a)) = \varphi_{\sigma,\eta}(\iota_n(a))$ for all $a \in A_n$.

Assumptions

Both H and K are complete complex Hilbert spaces. Only π is required to be irreducible with nonzero action. The representation σ is unital but may be reducible. The target vector has norm 1.

Conclusion

The two vector states agree exactly on the whole chosen matrix stage, not merely approximately on a finite list. The unit vector ξ may depend on the stage. No single vector realizing the target state on all of A is asserted.

Proof route

Encode the target stage restriction as a Gram matrix, realize that matrix in the root corner of π, and reconstruct a vector by applying the first-column matrix units. Matrix-unit coefficients determine a linear functional on a full matrix algebra, and evaluation at the unit gives normalization.

Proof steps

1. Set $v_i = \sigma(e_{0i}^{(n)})\eta$. The matrix-unit relations and the star property give $\langle v_i, v_j\rangle = \varphi_{\sigma,\eta}(e_{ij}^{(n)})$. Use the root-corner Gram realization theorem to choose $w_i \in \operatorname{ran} \pi(e_{00}^{(n)})$ with the same inner products.
2. Define $\xi = \sum_i \pi(e_{i0}^{(n)})w_i$. Multiplication by a matrix unit gives $\pi(e_{pq}^{(n)})\xi = \pi(e_{p0}^{(n)})w_q$. Taking the inner product with ξ yields $\varphi_{\pi,\xi}(e_{pq}^{(n)}) = \langle w_p, w_q\rangle$.
3. The two vector functionals therefore agree on every matrix unit, and linearity extends equality to every element of A_n. At its identity, the target value is $\|\eta\|^2 = 1$ and the reconstructed value is $\|\xi\|^2$, so $\|\xi\| = 1$.

Main citations

- The stated existence or structural result · Exact source
- Gram realization in the root corner · Exact source
- Reconstruction from root-corner vectors · Exact source
- Matrix-unit action on the reconstructed vector · Exact source
- Its matrix-unit coefficients · Exact source
- Matrix units determine a stage functional · Exact source
- Equality on the whole stage · Exact source
- Normalization of the reconstructed vector · Exact source
- The infinite-dimensional corner supplying room · Exact source
- Representation and nonzero irreducibility conventions · Exact source

Lean source signature (exact)

```
theorem exists_unitVector_vectorFunctional_eq_on_stage
    {H K : Type*}
    [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
    [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
    (ρ : Representation Limit H) (hρ : ρ.IsIrreducible)
    (σ : Representation Limit K) (n : ℕ) (η : K) (hη : ‖η‖ = 1) :
    ∃ ξ : H, ‖ξ‖ = 1 ∧ ∀ c : Stage n,
      Representation.vectorFunctional ρ ξ (ofStage n c) =
        Representation.vectorFunctional σ η (ofStage n c)
```

> Here ρ is π, σ is σ, and `Representation.vectorFunctional` is φ. The input η is the target unit vector and the existential ξ is its finite-stage realization. `ofStage n a` denotes $\iota_n(a)$. The proof uses `reconstructedStageVector` for the finite sum in the second step.

Lean realization notes

Finite-stage purification is compatible with mixed restrictions of the target state: no purity assumption on that restriction is needed. The extra room comes from the infinite-dimensional represented root corner.

▼ **Exact Card identity**

Language: en

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