Closure spaces before topology

`ClosureSpace`

structure

Closure spaces before topology

Statement

A closure space is a set with an extensive, monotone and idempotent operation on its subsets that sends the empty set to the empty set. Preservation of finite unions is a separate condition.

Definition

Let X be a set and let $\mathcal{P}(X)$ denote its power set, ordered by inclusion. A closure space on X consists of a map $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying, for all $A, B \subseteq X$,

$$\begin{aligned} A &\subseteq c(A), \\ A \subseteq B &\Longrightarrow c(A) \subseteq c(B), \\ c(c(A)) &= c(A), \\ c(\varnothing) &= \varnothing. \end{aligned}$$

A subset A is **closed** when $c(A) = A$. The first three axioms say that c is a closure operator on the ordered set $\mathcal{P}(X)$; the fourth supplies the empty-set condition.

The cited intersection theorem gives the useful equivalent description

$$c(A) = \bigcap \{ B \subseteq X : A \subseteq B \text{ and } c(B) = B \}.$$

Indeed, $c(A)$ is closed and contains A, and monotonicity puts it inside every closed superset of A. Thus $c(A)$ is the smallest closed set containing A.

Assumptions

The underlying set X is arbitrary and may be empty. To specify a closure space, one supplies the map c and the four displayed properties. No topology, metric or separation condition on X is required.

Conclusion

The resulting structure provides a closure operation and a notion of closed subset, with arbitrary intersections of closed sets again closed. A topological closure requires the additional identity

$$c(A \cup B) = c(A) \cup c(B).$$

This extra condition is called topologizability in the cited source. It is not one of the four axioms above.

Main citations

- Exact structure and closure laws: `ClosureSpace`, lines 28–30; its operation and laws, lines 37–61.
- Closed intersections and closure minimality: `ClosureSpace.closure_min`, `ClosureSpace.isClosed_sInter` and `ClosureSpace.closure_eq_sInter`, lines 78–99.
- The additional finite-union condition: `ClosureSpace.IsTopologizable`, lines 102–103.
- The underlying order-theoretic structure: Mathlib `ClosureOperator` in `Order/Closure.lean`, at the pinned mathlib commit.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 3.1 and Remark 3.4: the four axioms and the intersection description.

Lean source signature (exact)

```
structure ClosureSpace (α : Type u) where
  toClosureOperator : ClosureOperator (Set α)
  map_empty' : toClosureOperator ∅ = ∅
```

In the source	Mathematical meaning
Exact declaration `α : Type u, Set α`	An arbitrary set X and its power set $\mathcal{P}(X)$; neither finiteness nor nonemptiness is required.
Exact declaration `toClosureOperator : ClosureOperator (Set α)`	The operation $c : \mathcal{P}(X) \to \mathcal{P}(X)$, with $A \subseteq c(A)$, $A \subseteq B \Rightarrow c(A) \subseteq c(B)$, and $c(c(A)) = c(A)$ for all subsets A, B. Its closed-set predicate agrees with $c(A) = A$.
Exact declaration `map_empty'`	The additional equality $c(\varnothing) = \varnothing$.
Exact declaration `ClosureSpace α`	The returned structure consists of this operation and all four laws. It does not include preservation of finite unions.

Exact declaration for the source guide

```
structure ClosureSpace (α : Type u) where
  toClosureOperator : ClosureOperator (Set α)
  map_empty' : toClosureOperator ∅ = ∅

namespace ClosureSpace

variable {α : Type u}

/-- The underlying set closure. -/
```

ClosureOperator · fixed Mathlib definition

Fixed source, lines 60–70 · Raw UTF-8 source

```
structure ClosureOperator [Preorder α] extends α →o α where
  /-- An element is less than or equal its closure -/
  le_closure' : ∀ x, x ≤ toFun x
  /-- Closures are idempotent -/
  idempotent' : ∀ x, toFun (toFun x) = toFun x
  /-- Predicate for an element to be closed.

  By default, this is defined as `c.IsClosed x := (c x = x)` (see `isClosed_iff`).
  We allow an override to fix definitional equalities. -/
  IsClosed (x : α) : Prop := toFun x = x
  isClosed_iff {x : α} : IsClosed x ↔ toFun x = x := by aesop
```