A full homeomorphism to a compact-operator GSS

`ClosureSpace.HasCompactOperatorHomeomorph`

def

Requires a full homeomorphism to the geometric structure space of a compact-operator algebra.

Statement

Let (X, c) be a closure space. The compact-operator homeomorphism condition holds when (X, c) is fully homeomorphic to $S(\mathcal{K}(H))$ for some nonzero complex Hilbert space H.

Definition

Here $\mathcal{K}(H)$ denotes the algebra of compact operators on H. For a complex normed space D, $S(D)$ consists of the unions of kernels of norm-one bounded complex-linear functionals supporting each maximal proper face of its closed unit ball. Faces use real convexity. The support-functional norm theorem justifies the norm-one description for these nonempty faces. Its closure is $c_D(\mathcal{U}) = \{I \in S(D) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$.

The condition requires a nonzero complex Hilbert space H and a closure-space homeomorphism

$$e : (X, c) \longrightarrow S(\mathcal{K}(H)).$$

This is a bijection with full continuity in both directions. Explicitly, the required inclusions are $e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E))$ for every $E \subseteq X$, and $e^{-1}(c_{\mathcal{K}(H)}(F)) \subseteq c(e^{-1}(F))$ for every target subset F. The inverse is the inverse of the same bijection. Existence is required; no preferred homeomorphism is selected.

Assumptions

Only the input closure space is given. The Hilbert space and its structures are existential data. Nonzero H is required; separability and finite dimension are not. The definition uses arbitrary-set closure and does not replace it by finite closure.

Conclusion

This condition requires a full geometric model, with both directions referring to the same bijection. A continuous finite comparison is a separate condition; it does not contain the inverse full-continuity field required here. No statement about an irreducible representation image is included in this definition.

Main citations

- Exact definition.

- Continuous finite compact comparison.

Lean source signature (exact)

Exact source and declaration. The complete defining right-hand side is included.

```
def HasCompactOperatorHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          Nonempty (Homeomorph c
            (NormedSpace.geometricStructureClosure ℂ
              (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

In the source	Mathematical meaning

Input closure
`def HasCompactOperatorHomeomorph.{uH, uX} {X : Type uX}`
`(c : ClosureSpace X) : Prop :=`
	The input c is the closure on X. The following existential statement defines the requested model condition; it does not select a preferred model or map.

One normed model space
`∃ (H : Type uH) (instN : NormedAddCommGroup H),`
`letI := instN`
	There must exist one normed additive group H. The first `letI` installs that chosen normed structure on H for the inner-product type that follows.

Complex inner product on that model
`∃ instI : InnerProductSpace ℂ H,`
`letI := instI`
	The same H must have a compatible complex inner product. This `letI` installs it for the later Hilbert-space and operator-algebra types.

Completeness of that model
`∃ instC : CompleteSpace H,`
`letI := instC`
	The same normed space must be complete. The next `letI` installs this completeness, used in the compact-operator algebra on this Hilbert space.

A nonzero model
`∃ instT : Nontrivial H,`
`letI := instT`
	The same H must be nonzero. Its `letI` makes this nontriviality available in the remaining conditions; it does not introduce another space.

Existence of a homeomorphism to the same model
`Nonempty (Homeomorph c`
`(NormedSpace.geometricStructureClosure ℂ`
`(ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`
	`Nonempty` requires an element of the displayed homeomorphism type: one bijection $$e : X \to S(\mathcal{K}(H))$$ with continuity in both directions. The complete target argument is the geometric closure $c_{\mathcal{K}(H)}$ on the compact-operator algebra of the same H used in all preceding conditions.

In the source	Mathematical meaning

The homeomorphism and its inverse
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The inherited equivalence gives e and its inverse e^{-1}. For every $E \subseteq X$ and $F \subseteq S(\mathcal{K}(H))$, the two fields require, respectively,
	$$e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E)),$$
	$$e^{-1}(c_{\mathcal{K}(H)}(F)) \subseteq c(e^{-1}(F)).$$
	Both use the same model and the same map; the definition does not choose separate forward and inverse maps.

Continuity for all subsets
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	For $r : (Y, d) \to (Z, k)$, the definition requires
	$$r(d(E)) \subseteq k(r(E)) \quad (E \subseteq Y).$$
	There is no finiteness restriction in this condition.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
namespace ClosureSpace

open MathlibAnnex.Analysis.CStarAlgebra
attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
```

The model-space level `uH` and the input-space level `uX` are independent fixed parameters. All model structures and conditions above refer to the same quantified H.

Related exact source for the reading guide

Continuity for all subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The homeomorphism and its inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```