Primitive representation kernels are exactly geometric component intersections

`MathlibAnnex.Analysis.CStarAlgebra.primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily`

theorem

Identifies primitive representation kernels with intersections of geometric components.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. A pure state is a positive norm-one bounded complex-linear functional on A that is extreme in the state space for real convexity. For a pure state φ, let π_φ be the representation of A obtained by restricting the GNS representation of its canonical state extension to the forced unitization. The independently defined primitive-kernel family is

$$\mathrm{Prim}(A) = \{\ker \pi_\varphi : \varphi \text{ is a pure state of } A\}.$$

Let $S(A)$ be the geometric structure space, whose points are kernels $\ker \rho \subseteq A$ of bounded complex-linear functionals $\rho : A \to \mathbb{C}$ that are extreme points, for real convexity, of the entire closed dual unit ball B_{A^*} by the extreme-kernel theorem, with the hull–kernel closure $c_A(\mathcal{T}) = \{I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I\}$. Let $\mathrm{GS}(A)$ be its finite-chain components (chains of equal points or nonclosed pairs), and put $J_C = \bigcap_{I \in C} I$. The primitive geometric family is $\mathrm{PS}(A) = \{J_C : C \in \mathrm{GS}(A)\}$. Then

$$\mathrm{Prim}(A) = \mathrm{PS}(A). \tag{1}$$

Here both $\mathrm{Prim}(A)$ and $\mathrm{PS}(A)$ are treated as families of subsets of A.

Assumptions

The source uses a nonzero complex C*-algebra with its compatible order. The family $\mathrm{Prim}(A)$ is constructed from the pure-state GNS representations independently of geometric components. The compared objects are families of subsets of this same algebra; no additional representation or component is an input.

Conclusion

The equality identifies precisely which subsets occur. It does not identify an individual component with its intersection. No injectivity of the representation-kernel map or of $C \mapsto J_C$ is asserted. The primitive-spectrum homeomorphism is a separate construction using this family equality.

Main citations

- Exact family equality: `primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily`.

- Pure-state component representative and pure-state component intersection: the related representative and same-state intersection statements.
- Independent primitive family and geometric component-intersection family: the definitions compared here.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily :
    primitiveKernelFamily A = NormedSpace.primitiveGeometricSpectrumFamily ℂ A
```

In the source	Mathematical meaning
Exact declaration `(A : Type u)` and its nonunital C*-algebra, order and nontriviality instances	The same nonzero complex C*-algebra underlies both independently defined families. No extra representation is an input to this declaration.
Exact declaration `primitiveKernelFamily A`	Kernels $\ker \pi_\varphi$ for pure states φ, forming the family $\mathrm{Prim}(A)$ of subsets of A.
Exact declaration `NormedSpace.primitiveGeometricSpectrumFamily ℂ A`	The family $\mathrm{PS}(A)$ of intersections of all GSS points in a component. It is not the family of the component subsets themselves.
The complete equality **Exact declaration**	Every member of either family is a member of the other, as the same subset of A. This compares representation kernels with whole-component intersections, not components with individual functional kernels.

Exact surrounding source context (separate excerpts)

Source lines 27–27:

```
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```

Source lines 28–28:

```
variable [Nontrivial A]
```

Exact declaration for the source guide

```
theorem primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily :
    primitiveKernelFamily A = NormedSpace.primitiveGeometricSpectrumFamily ℂ A := by
```