Preimages of every compact operator

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_preimage_of_compact_singleton`

Establishes the inclusion K(H) ⊆ π(A) by giving a preimage for every compact operator.

Statement

Let A be a nonzero complex C*-algebra, not assumed unital; let H be a separable complex Hilbert space; let π be a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A; and let T be a compact bounded operator on H. There exists a∈A with π(a)=T. This is the inclusion K(H)⊆π(A); compactness of each π(a) is proved separately.

Assumptions

Let A be a nonzero complex C*-algebra, not assumed unital; let H be a separable complex Hilbert space; let π be a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A; and let T be a compact bounded operator on H.

Conclusion

There exists a∈A with π(a)=T. This is the inclusion K(H)⊆π(A); compactness of each π(a) is proved separately.

Proof idea

The source combines injectivity of π with the preimages of every rank-one operator, then applies the closed-range/compact-operator generation lemma to obtain an exact preimage of T.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : NonUnitalCStarAlgebra A] {H : Type v} [inst_1 : NormedAddCommGroup H] [inst_2 : InnerProductSpace ℂ H] [inst_3 : CompleteSpace H] [Nontrivial A] [inst_5 : PartialOrder A] [StarOrderedRing A] [TopologicalSpace.SeparableSpace H] (pi : MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation A H), pi.IsSingletonIrreducibleModel → ∀ (T : H →L[ℂ] H), IsCompactOperator ⇑T → ∃ a, pi a = T

2. IsCompactOperator T → ∃ a : A, π a = T.

3. The source combines injectivity of π with the preimages of every rank-one operator, then applies the closed-range/compact-operator generation lemma to obtain an exact preimage of T.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_preimage_rankOne_of_singleton

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem exists_preimage_of_compact_singleton [Nontrivial A]
[PartialOrder A] [StarOrderedRing A]
[TopologicalSpace.SeparableSpace H]
(pi : NonUnitalCStarRepresentation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi)
(T : H →L[ℂ] H) (hT : IsCompactOperator T) :
∃ a : A, pi a = T
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: 0edd904a8f54cb2f656607289f4094b52459b64013e28ed399e00bc0acd454d8

Card SHA-256: ef9634a120bc677817374b923a401e6b48677300dcbed8bfe81cb3c9b001d2c0

Approved exposition SHA-256: f6c0409a3550536d77312da557022f12c334dc45ce9eda9d8af5f106baa8dfed

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.