The cyclic sum fills every irreducible target representation

`MathlibAnnex.CStarAlgebra.CAR.exists_surjective_selectedAtomicCyclicIsometry`

theorem

Uses shell reconstruction and residual lines to turn an isometric CAR intertwiner into a unitary equivalence on the source.

Statement

Under the shell-family, generator and irreducibility assumptions below, there exist unit vectors $\zeta \in K$, $\eta_j \in K$ for $j \in I$, and a surjective complex-linear isometry $W : \mathcal{H} \to K$ such that

$$W(e_j\xi_j) = \eta_j, \qquad V_j\eta_j = \zeta$$

for every j, and $W\pi(a) = \sigma(a)W$ for every $a \in A$. In particular, $\sigma = \rho \circ \iota$ is unitarily equivalent to the selected atomic source representation π.

Assumptions

Let A be the completed CAR algebra, the completion of the matrix stages $M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$. Let f_n be the image of the first diagonal matrix unit, so $f_0 = 1$ and f_n decreases through projections. The root state ω takes the value a_{00} on each matrix stage.

Choose one pure state φ_j from each unitary-equivalence class $j \in I$ of pure-state Gelfand-Naimark-Segal (GNS) representations, with root index o and $\varphi_o = \omega$. Write (H_j, π_j, ξ_j) for its complete complex GNS space, unital star representation and unit cyclic vector. Set $\mathcal{H} = \bigoplus_{j\in I} H_j$, $\pi = \bigoplus_j \pi_j$, and let $e_j : H_j \to \mathcal{H}$ be the coordinate embedding. No countability of I is assumed. Inner products are linear in the second argument.

A supplied shell family consists of unital complex star automorphisms α_j and elements $w_{j,n} \in A$. Put $f_{j,n} = \alpha_j(f_n)$. The identities are $\varphi_j \circ \alpha_j = \omega$, $w_{j,n}^* w_{j,n} = f_{j,n} - f_{j,n+1}$ and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$. The family is an input, not an existence conclusion.

Let L_j be unitaries on $\mathcal{H}$ satisfying $L_j\pi(f_{j,n} - f_{j,n+1}) = \pi(w_{j,n})$ for all j, n, and assume $L_o = I_{\mathcal{H}}$. Let $T = C^*(\pi(A), \{L_j\}) \subseteq B(\mathcal{H})$ be the norm-closed unital star algebra they generate. Write $\iota(a) = \pi(a)$ as an element of T, and $g_j \in T$ for the element represented by L_j. Let $\rho : T \to B(K)$ be a unital complex star representation on a complete complex Hilbert space, put $\sigma = \rho \circ \iota$, and put $V_j = \rho(g_j)$. Assume ρ is nonzero and irreducible: its only closed reducing subspaces are $\{0\}$ and K. This theorem does not assume the additional identity $L_j(e_j\xi_j) = e_o\xi_o$. The root normalization $L_o = I_{\mathcal{H}}$ is required.

Conclusion

The selected cyclic pieces exhaust the target Hilbert space. The source-restriction equivalence is the direct consequence obtained by viewing the surjective isometry as a unitary. The representation being assumed irreducible is ρ on T, not its generally reducible restriction σ on A.

Proof route

The common-root vector-state theorem supplies compatible unit vectors. Cyclic-sum assembly gives an isometry with closed source-reducing range M. Shell reconstruction splits each represented generator into a strong shell sum and a residual corner. Both parts and their adjoints preserve M. Norm-closed generation then makes M a nonzero reducing subspace for ρ, so irreducibility forces $M = K$.

Proof steps

1. The common-root construction gives ζ and η_j of norm one with the selected CAR vector states, $V_j\eta_j = \zeta$, and fixedness under their respective flags. Assemble the cyclic pieces into W. Its range M is closed because $\mathcal{H}$ is complete and W is an isometry. The relations for a and a^* show that M reduces σ. Let P_j project onto the common range of $\sigma(f_{j,n})$ and let Q project onto the common range of $\sigma(f_n)$. Assembly gives $P_jM \subseteq \mathbb{C}\eta_j$.
2. Since $L_o = I_{\mathcal{H}}$, the represented root generator is $V_o = I_K$. Thus the root-index member η_o equals the common vector ζ. Consequently $\zeta \in M$ and $QM \subseteq \mathbb{C}\zeta$. Reconstruct $V_j = S_j + R_j$ with $R_j = V_jP_j$, $R_j^*R_j = P_j$, $R_jR_j^* = Q$ and $R_j = QR_jP_j$. The operators S_j and S_j^* are strong limits of finite sums of $\sigma(w_{j,n})$ and their adjoints. Closedness and source reduction make M invariant under both limits.
3. Fixedness and the vector transport give $R_j\eta_j = \zeta$ and $R_j^*\zeta = \eta_j$. For $y \in M$, the vector P_jy lies in $\mathbb{C}\eta_j$, so $R_jy \in \mathbb{C}\zeta \subseteq M$. For the adjoint, use $R_j^* = P_jR_j^*Q$ and $Qy \in \mathbb{C}\zeta$ to obtain $R_j^*y \in \mathbb{C}\eta_j \subseteq M$. Thus M reduces the residual corner and, together with the shell sum, every V_j.
4. The orthogonal projection onto M commutes with the represented source and every added generator. The commutation relation is stable under star-algebra operations and operator-norm limits, so M reduces all of $\rho(T)$. It contains the unit vector ζ and is therefore nonzero. Irreducibility gives $M = K$, which is exactly surjectivity of W. Its pointed-vector and source intertwining identities are retained.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedAtomicRepresentation · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_targetDefectVectorFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_selectedAtomicCyclicIsometry · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_targetShellReconstruction · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.reduces_concreteTarget_of_generators · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedAtomicRepresentation_unitaryEquivalent_restricted · Exact source
- MathlibAnnex.CStarAlgebra.CAR.ambientInclusion_unitaryEquivalent · Exact source
- MathlibAnnex.CStarAlgebra.CAR.completedRootPureState · Exact source

Lean source signature (exact)

```
theorem exists_surjective_selectedAtomicCyclicIsometry
    (family : RepresentativeShellFamily)
    (L : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit →
      MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState)
    (hLunit : ∀ i, L i ∈ unitary
      (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState))
    (hLsource : ∀ i n, (L i).comp (selectedAtomicRepresentation
      (transportedFlag family i n - transportedFlag family i (n + 1))) =
        representedShellLink family i n)
    (hLroot : L completedRootPureState.classOf = 1)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K] (rho : Representation (AtomicTarget L) K)
    (hrho : rho.IsIrreducible) :
    ∃ (eta_o : K) (eta : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit → K)
      (W : MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →ₗᵢ[ℂ] K),
      Function.Surjective W ∧
      ‖eta_o‖ = 1 ∧
      (∀ i, ‖eta i‖ = 1) ∧
      (∀ i, W (MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState i
        (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState i)) = eta i) ∧
      (∀ i, (Unitary.linearIsometryEquiv
        (representedGeneratorUnitary L hLunit rho i) : K ≃ₗᵢ[ℂ] K)
          (eta i) = eta_o) ∧
      ∀ a,
        W.toContinuousLinearMap.comp (selectedAtomicRepresentation a) =
          ((restrictedRepresentation L rho) a).comp
            W.toContinuousLinearMap
```

> Here `Limit` is A, `completedRootPureState` is the root state ω bundled with its purity proof, and `SelectedAtomicHilbert ...` is $\mathcal{H}$. `AtomicTarget L` is T and `restrictedRepresentation L rho` is σ. Source `eta_o` denotes the common vector ζ; `eta i` denotes η_j at source index `i`. The equality of `eta` at the root index with `eta_o` follows from `hLroot`, rather than from their names. `Function.Surjective W` supplies the extra property that the isometry fills K. `hLsource` records the shell action; no hypothesis named `hLmap` occurs in this declaration.

Lean realization notes

The conclusion displayed here intertwines the CAR source. To obtain intertwining for every element of T, the full capture theorem also uses the prescribed action $L_j(e_j\xi_j) = e_o\xi_o$. No multiplicity assertion for arbitrary reducible target representations and no unconditional existence of a shell family is claimed.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
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Card revision: 1 · SHA-256: 5fb3870842607637f36f0d394546e845aa17652f3b2b9f5363bebcc9c2a4d26d
Exposition revision: 1 · SHA-256: 8d9b9d22f8e6fca5a6f830d653607771e8f448c6d887a8c23a1277b891e9f59b
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
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