Simplicity from a faithful representative of the unique irreducible-representation class

`MathlibAnnex.Analysis.CStarAlgebra.Representation.isSimpleCStarAlgebra_of_singleton_of_injective`

Converts a faithful representative of the unique irreducible-representation class into closed-two-sided-ideal simplicity of the algebra.

Statement

Let A be a nonzero unital complex C*-algebra, H a complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. Assume additionally that π is injective. No separability assumption on H is required. A is simple in the exact closed-two-sided-ideal sense IsSimpleCStarAlgebra A.

Assumptions

Let A be a nonzero unital complex C*-algebra, H a complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. Assume additionally that π is injective. No separability assumption on H is required.

Conclusion

A is simple in the exact closed-two-sided-ideal sense IsSimpleCStarAlgebra A.

Proof idea

For a closed two-sided ideal I, the source splits I=⊤ from a proper I. In the proper case an irreducible GNS representation annihilating I belongs to the unique class represented by π; the intertwiner makes π vanish on each x∈I, and injectivity forces x=0, so I=⊥.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : CStarAlgebra A] [inst_1 : PartialOrder A] [StarOrderedRing A] {H : Type v} [inst_3 : NormedAddCommGroup H] [inst_4 : InnerProductSpace ℂ H] [inst_5 : CompleteSpace H] [Nontrivial A] (pi : MathlibAnnex.Analysis.CStarAlgebra.Representation A H), pi.IsSingletonIrreducibleModel → Function.Injective ⇑pi → MathlibAnnex.Analysis.CStarAlgebra.IsSimpleCStarAlgebra A

2. IsSingletonIrreducibleModel π ∧ Injective π → IsSimpleCStarAlgebra A.

3. For a closed two-sided ideal I, the source splits I=⊤ from a proper I. In the proper case an irreducible GNS representation annihilating I belongs to the unique class represented by π; the intertwiner makes π vanish on each x∈I, and injectivity forces x=0, so I=⊥.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.isIrreducible_pureState_gnsStarAlgHom

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem isSimpleCStarAlgebra_of_singleton_of_injective [Nontrivial A]
(pi : Representation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi)
(hinj : Function.Injective pi) : IsSimpleCStarAlgebra A
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: 106c2e5861a90efa2b3d1125a38099aaae4192ca2a311076da995a4ae3043e03

Card SHA-256: 3e840be62b2cd37be2177465ee59017e1bba2f722c38e1e8234b5ca2f888b123

Approved exposition SHA-256: a311f0b148c937aac720c808da3c3570ce6cb520f8a1b62953e21141de20a478

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