Preimages for all operators of the form rankOne

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_preimage_rankOne_of_singleton`

Provides preimages for every operator of the form rankOne ℂ x y, the rank-at-most-one generators used to obtain all compact operators.

Statement

Let A be a nonzero complex C*-algebra, not assumed unital; let H be a separable complex Hilbert space; and let π be a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. For every pair x,y∈H, including zero vectors, there exists a∈A with π(a)=rankOne ℂ x y; this operator has rank at most one.

Assumptions

Let A be a nonzero complex C*-algebra, not assumed unital; let H be a separable complex Hilbert space; and let π be a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A.

Conclusion

For every pair x,y∈H, including zero vectors, there exists a∈A with π(a)=rankOne ℂ x y; this operator has rank at most one.

Proof idea

The source obtains one nonzero rank-one projection in π(A), uses the unique-class hypothesis for π together with injectivity, and invokes the rank-one-preimage theorem to move from that projection to every x and y, including zero vectors.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : NonUnitalCStarAlgebra A] {H : Type v} [inst_1 : NormedAddCommGroup H] [inst_2 : InnerProductSpace ℂ H] [inst_3 : CompleteSpace H] [Nontrivial A] [inst_5 : PartialOrder A] [StarOrderedRing A] [TopologicalSpace.SeparableSpace H] (pi : MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation A H), pi.IsSingletonIrreducibleModel → ∀ (x y : H), ∃ a, pi a = ((InnerProductSpace.rankOne ℂ) x) y

2. ∀ x y : H, ∃ a : A, π a = rankOne ℂ x y.

3. The source obtains one nonzero rank-one projection in π(A), uses the unique-class hypothesis for π together with injectivity, and invokes the rank-one-preimage theorem to move from that projection to every x and y, including zero vectors.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_nonzero_projection_rankOne_map

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

```
theorem exists_preimage_rankOne_of_singleton [Nontrivial A]
[PartialOrder A] [StarOrderedRing A]
[TopologicalSpace.SeparableSpace H]
(pi : NonUnitalCStarRepresentation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi) :
∀ x y : H, ∃ a : A, pi a = InnerProductSpace.rankOne ℂ x y
```

Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: 155676cd74606a102aa274be40c9fb294971cfef6df06897084c9fc89322f2cb

Card SHA-256: a1c2ac8a3243c4697c1d5133870dfbda664c5a4e32cfba5f403fe5d6f387df5a

Approved exposition SHA-256: 292d3d95418d0c7dc1e14e45463b44bfe2a2ae1074cb7938ad66e19e10ec44f4

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.