Faithful normal-part transport through a specified GSS homeomorphism

`MathlibAnnex.Analysis.CStarAlgebra.exists_faithful_normalPart_transport`

theorem

Transports the normal part through one faithful pure-GNS representation and the same preimage map.

Statement

Let H be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a nonzero norm-closed complex $*$-subalgebra acting irreducibly on H. Let B be a complex C*-algebra with its compatible order. Write $S(D)$ for the geometric structure space of an algebra D, with hull–kernel closure c_D. For each maximal proper face of the closed unit ball of D, take the union of the kernels of all bounded complex-linear functionals of norm one that equal 1 on that face. The distinct subsets so obtained are the points of $S(D)$, by the geometric structure definition and the norm-one support characterization.

For a concrete algebra $D \subseteq \mathcal{B}(L)$, define

$$f^D_{x,y} : D \to \mathbb{C}, \qquad f^D_{x,y}(a) = \langle y, a(x) \rangle, \qquad x, y \in L \setminus \{0\},$$

and

$$N(D) = \{I \in S(D) : I = \ker f^D_{x,y} \text{ for some } x, y \in L \setminus \{0\}\}.$$

The inner product is linear in the second variable. Suppose that $e : S(A) \to S(B)$ is a homeomorphism of closure spaces, so that

$$e(c_A(E)) = c_B(e(E)) \qquad (E \subseteq S(A)).$$

Then there are a pure state φ of B and a homeomorphism

$$r : S(\pi_\varphi(B)) \longrightarrow S(B),$$

where $\pi_\varphi : B \to \mathcal{B}(H_\varphi)$ is the GNS representation of this same state, with the following properties:

- π_φ is faithful.
- For its range restriction $\pi_{\varphi,r} : B \to \pi_\varphi(B)$,

$$r(J) = \pi^{-1}_{\varphi,r}(J) \qquad (J \in S(\pi_\varphi(B))). \qquad (1)$$

- The inverse of that same homeomorphism satisfies

$$\pi_{\varphi,r}^{-1}(r^{-1}(I)) = I \qquad (I \in S(B)). \qquad (2)$$

- The normal parts satisfy

$$(r^{-1} \circ e)(N(A)) = N(\pi_\varphi(B)). \qquad (3)$$

Assumptions

The Hilbert spaces are complete. A pure state is a positive bounded complex-linear functional of norm one that is extreme in the state space for real convexity. Closedness and irreducibility concern the given action of A on H. The homeomorphism e is specified, with continuity for all subsets in both directions. Nonzero H and B follow from these hypotheses; no extra nontriviality assumption on B is imposed.

Conclusion

One faithful pure-GNS representation and one inverse-image homeomorphism satisfy all three formulas. In (3), the same map e is followed by r^{-1}. The conclusion identifies the normal parts under this composite; it does not identify a normal part with its entire geometric structure space.

Main citations

Faithful normal-part transport gives all the outputs for the same pure state and the same homeomorphism. The component-density criterion characterizes faithfulness of the pure-GNS representation. The forward preimage formula and inverse preimage formula concern the same range restriction; the normal-part image identity specifies which subset is transported.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem exists_faithful_normalPart_transport
    (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))]
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (e : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ S)
      (NormedSpace.geometricStructureClosure ℂ B)) :
    ∃ phi : NonUnitalPureState B, Function.Injective phi.gnsRepresentation ∧
      ∃ r : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ
(NonUnitalStarAlgHom.range phi.gnsRepresentation))
          (NormedSpace.geometricStructureClosure ℂ B),
        (∀ J, (r.toEquiv J).val = (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹'
J.val) ∧
        (∀ I, (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹' (r.toEquiv.symm
I).val = I.val) ∧
        (e.trans r.symm).toEquiv '' normalPart S = normalPart (NonUnitalStarAlgHom.range
phi.gnsRepresentation)
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning
Original Hilbert space `variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`	The original representation space H is a complete complex Hilbert space. It is independent of the GNS space of the state found in the conclusion.
Target algebra `variable {B : Type w} [NonUnitalCStarAlgebra B] variable [PartialOrder B] [StarOrderedRing B]`	The complex C*-algebra B has its compatible order. No separate nontriviality assumption on B appears here.
Concrete irreducible action `(S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))] (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`	The source name S is the closed complex $*$-subalgebra $A \subseteq \mathcal{B}(H)$. Its inclusion action is nonzero and irreducible.
Given homeomorphism `(e : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ S) (NormedSpace.geometricStructureClosure ℂ B)) :`	The input is the specified closure-space homeomorphism $e : S(A) \to S(B)$. Its domain and codomain use the ambient geometric closures, not closures restricted to normal parts.
One faithful pure-state representation `∃ phi : NonUnitalPureState B, Function.Injective phi.gnsRepresentation ∧`	There exists one pure state φ of B whose GNS representation π_φ is injective, hence faithful. This same state and representation occur in every subsequent condition.
Homeomorphism from the representation image `∃ r : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ (NonUnitalStarAlgHom.range phi.gnsRepresentation)) (NormedSpace.geometricStructureClosure ℂ B),`	There exists one homeomorphism $r : S(\pi_\varphi(B)) \to S(B)$. The range is the actual image algebra of this same representation, not all bounded operators on its GNS space.
Forward preimage formula `(∀ J, (r.toEquiv J).val = (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹' J.val) ∧`	The underlying bijection of r sends every geometric point J of the image algebra to $$r(J) = \pi_{\varphi,r}^{-1}(J).$$ Here $\pi_{\varphi,r} : B \to \pi_\varphi(B)$ is the range restriction. The projections expose the underlying subsets.
Inverse preimage formula `(∀ I, (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹' (r.toEquiv.symm I).val = I.val) ∧`	The inverse of that same bijection satisfies $$\pi_{\varphi,r}^{-1}(r^{-1}(I)) = I \qquad (I \in S(B)).$$ The inverse is not chosen independently.
Transport of normal parts `(e.trans r.symm).toEquiv '' normalPart S = normalPart (NonUnitalStarAlgHom.range phi.gnsRepresentation) := by`	Transitivity composes e first and r^{-1} second. Taking its underlying bijection and its set image gives $$(r^{-1} \circ e)(N(A)) = N(\pi_\varphi(B)).$$ This identifies the normal parts, not the whole geometric structure spaces.

In the source	Mathematical meaning

Same bijection in both continuity fields
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	For a homeomorphism $q : (X, c_X) \to (Y, c_Y)$, the underlying equivalence supplies q and its inverse. The fields assert
	$$q(c_X(E)) \subseteq c_Y(q(E)), \qquad E \subseteq X,$$
	$$q^{-1}(c_Y(F)) \subseteq c_X(q^{-1}(F)), \qquad F \subseteq Y.$$
	This applies to both e and r with their displayed domains.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]

attribute [local instance] HomKernelQuotient.imageCStarAlgebra HomKernelQuotient.isClosed_image
```

Exact surrounding source:

```
variable {B : Type w} [NonUnitalCStarAlgebra B]
variable [PartialOrder B] [StarOrderedRing B]
```

Related exact source for the reading guide

Transport of normal parts · Fixed source

```
        (e.trans r.symm).toEquiv '' normalPart S = normalPart (NonUnitalStarAlgHom.range
phi.gnsRepresentation) := by
```

Same bijection in both continuity fields · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```