Distinct GNS classes have no unitary intertwiner

`MathlibAnnex.CStarAlgebra.PureState.no_unitaryIntertwiner_selectedRepresentation`

theorem

The class index makes the selected pure-GNS family pairwise unitarily inequivalent.

Statement

Let A be a unital complex C*-algebra, with the positive order used for its states, and fix a pure state ω. A pure state is an extreme point of the normalized positive continuous linear functionals. Let I be the set of pure-state GNS representations modulo unitary equivalence. For each $j \in I$, choose a pure state φ_j in that class, retaining $\varphi_o = \omega$ literally at $o = [\omega]$. Write (H_j, π_j, ξ_j) for its GNS space, representation and canonical cyclic vector. If $i \neq j$, no surjective complex-linear isometry $E : H_i \to H_j$ intertwines the two selected representations. Explicitly, for every such E, the equations $E\pi_i(a) = \pi_j(a)E$ cannot hold for all $a \in A$.

Assumptions

The GNS classes $i, j \in I$ are assumed to be distinct. A candidate E is a unitary map between the two complete Hilbert spaces; it need not carry ξ_i to ξ_j. The same representative selection based at ω is used on both sides.

Conclusion

The selected representations form a family with no unitary intertwiners between distinct indices. This supplies precisely the pairwise-inequivalence hypothesis of the generic atomic-shell construction.

Proof route

An intertwining unitary would certify GNS equivalence of the two selected states. Taking their quotient classes would therefore identify i and j, contradicting their assumed distinctness.

Proof steps

1. Assume that the candidate unitary E satisfies the intertwining equations. The selected representations are exactly the GNS actions of φ_i and φ_j, so E witnesses their GNS equivalence.

2. The cited representative-class theorem says that GNS-equivalent selected representatives have equal indices: each chosen representative has the class it was chosen from. Applying that theorem gives $i = j$, contrary to the hypothesis.

Main citations

- The stated existence or structural result · Exact source
- Selected concrete GNS representations · Exact source
- Equivalent selected representatives have the same class · Exact source
- Each representative lies in its specified class · Exact source

Lean source signature (exact)

```
theorem no_unitaryIntertwiner_selectedRepresentation (root : PureState A)
    {i j : GNSClass A} (hij : i ≠ j) (e : SelectedGNS root i ≃ₗᵢ[ℂ] SelectedGNS root j) :
    ¬ StarAlgHom.Intertwines (selectedRepresentation root i)
      (selectedRepresentation root j)
      (e : SelectedGNS root i →L[ℂ] SelectedGNS root j)
```

> Here `root` is ω and `hij` is $i \neq j$. The linear isometry equivalence `e` is the candidate E, so it is already surjective. `StarAlgHom.Intertwines` expresses $E\pi_i(a) = \pi_j(a)E$ for every a. In the linked proof, `representative_injective_on_classes` converts that unitary equivalence into equality of the class indices; it is separately cited.

Lean realization notes

Two distinct pure states can belong to the same GNS class. The assertion here is about distinct classes. Vanishing of arbitrary bounded intertwiners requires an additional irreducibility argument and is not the conclusion of this declaration.

▼ **Exact Card identity**

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Exposition revision: 1 · SHA-256: 557337b0dc01c1feef1dfba6501ca3f7f27f23f478b276fcf94d407c67d0a078
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