A singleton irreducible spectrum is witnessed by a faithful, fully normal representation

`MathlibAnnex.Analysis.CStarAlgebra.exists_forall_eq_irreducibleSpectrum_iff_exists_faithful_representation`

theorem

Characterizes a singleton irreducible spectrum by one faithful representation with a full normal part.

Statement

Let A be a nonzero complex C^*-algebra, not necessarily unital, with the compatible order structure used in its pure-state construction. Write $\widehat{A}$ for its irreducible spectrum: the space of unitary-equivalence classes of irreducible representations, realized here through pure-state GNS representations. It is not the space of primitive ideals obtained by identifying representations with the same kernel.

For a nonzero irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space H, write $S(\pi(A))$ for the geometric structure of its image $\pi(A)$, whose points are subsets obtained as unions of supporting kernels for maximal proper real faces of the closed unit ball. The established C^*-algebra identification realizes these points as kernels of extreme norm-one functionals. Use the inner product linear in its second variable. For $x, y \in H \setminus \{0\}$, define the bounded linear functional $f_{x,y} : \pi(A) \to \mathbb{C}$ by $f_{x,y}(t) = \langle y, t(x)\rangle$. Its normal part consists of these coefficient kernels

$$N(\pi(A)) = \{\ker f_{x,y} : x, y \in H \setminus \{0\}\}. \qquad (1)$$

The following are equivalent:

(i) The irreducible spectrum $\widehat{A}$ has exactly one point.

(ii) There are a complex Hilbert space H and a nonzero irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ such that

$$\ker \pi = \{0\}, \qquad N(\pi(A)) = S(\pi(A)).$$

The same Hilbert space and representation satisfy all the conditions in (ii).

Assumptions

The algebra is nontrivial. There is no separability, unitality, finite-dimensionality, or prescribed Hilbert space in the input. Irreducibility on the right includes a nonzero action; it is not a convention that permits the zero representation. The order and star-order structures in the ambient source are

retained. No full-normal-part or faithfulness assumption is hidden in a newly defined representation object.

Here “fully normal” means exactly the equality $N(\pi(A)) = S(\pi(A))$ of sets of geometric points in (ii). It is not a claim about normal functionals on a von Neumann algebra, nor about the ultraweak continuity of π.

Conclusion

A singleton unitary-equivalence spectrum is equivalent to the simultaneous existence of faithfulness and a full normal part for one nonzero irreducible representation. Neither condition alone replaces their conjunction. No elementary-algebra classification is asserted here.

Main citations

The singleton-spectrum representation theorem gives the equivalence of (i) and (ii). The pure-state witness theorem supplies one faithful GNS representation with full normal part; the converse for a given representation applies to the same representation in (ii).

Lean source signature (exact)

The preceding `variable` declarations supply the nonzero C*-algebra and its order assumptions. On the right of `↔`, the nested existentials choose one complex Hilbert space and one representation on it. The norm, inner product and completeness all belong to that same space; the local `letI` declarations make these structures available together. The three final conjuncts concern that same representation. The following contiguous prefix ends before the proof assignment.

```
theorem exists_forall_eq_irreducibleSpectrum_iff_exists_faithful_representation :
    (∃ q : IrreducibleSpectrum A, ∀ r : IrreducibleSpectrum A, r = q) ↔
    (∃ (H' : Type u) (instN : NormedAddCommGroup H'),
      letI := instN
      ∃ instI : InnerProductSpace ℂ H',
        letI := instI
        ∃ instC : CompleteSpace H',
          letI := instC
          ∃ pi : NonUnitalCStarRepresentation A H',
            pi.IsIrreducible ∧ Function.Injective pi ∧ normalPart (NonUnitalStarAlgHom.range
pi) = Set.univ)
```

In the source	Mathematical meaning

Nonzero algebra and compatible order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The input is the nonzero complex C*-algebra A. The partial order and star-order structure are those used for positivity and pure states. Unitality and separability are not required.

In the source	Mathematical meaning
A singleton irreducible spectrum `(∃ q : IrreducibleSpectrum A, ∀ r : IrreducibleSpectrum A, r = q) ↔`	The left side says that $\widehat{A}$ has exactly one point, including existence of that point. Its points are unitary-equivalence classes of irreducible representations; this is not a statement that their kernels alone coincide.
One Hilbert space and one representation `(∃ (H' : Type u) (instN : NormedAddCommGroup H'), letI := instN ∃ instI : InnerProductSpace ℂ H', letI := instI ∃ instC : CompleteSpace H', letI := instC ∃ pi : NonUnitalCStarRepresentation A H',`	One complex Hilbert space H is chosen, with its norm, compatible inner product and completeness. One $*$-representation $\pi : A \to \mathcal{B}(H)$ is then chosen on this same space. All conditions in the next row concern this same π.
Irreducibility, faithfulness, and the full normal part `pi.IsIrreducible ∧ Function.Injective pi ∧ normalPart (NonUnitalStarAlgHom.range pi) = Set.univ) := by`	The representation π is nonzero and irreducible, is faithful, and has full normal part: $$\ker \pi = \{0\},$$ $$N(\pi(A)) = S(\pi(A)).$$ The latter equality is with the whole geometric structure space of its image $\pi(A)$, not with the operator algebra $\mathcal{B}(H)$.
Coefficient kernels in the image algebra `def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) : Set (NormedSpace.GeometricStructureSpace ℂ S) := {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧ {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}`	Apply the definition to the image algebra $\pi(A)$. Its normal part consists of the coefficient kernels $$N(\pi(A)) = \{\ker f_{x,y} : x, y \in H \setminus \{0\}\},$$ where $f_{x,y} : \pi(A) \to \mathbb{C}$ is given by $f_{x,y}(t) = \langle y, t(x)\rangle$, as in (1). The kernel is identified with the underlying subset of the geometric point I.

Exact surrounding source context (separate excerpts)

SpectrumSingleton.lean:

```
universe u v
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

The existential Hilbert space has source level `u`, the same fixed level as A. The separate ambient level `v` does not enlarge this theorem's model quantifier.

Related exact source for the reading guide

Coefficient kernels in the image algebra · Fixed source

```
def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) :
    Set (NormedSpace.GeometricStructureSpace ℂ S) :=
  {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧
    {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}
```

Irreducibility, faithfulness, and the full normal part · Fixed source

```
            pi.IsIrreducible ∧ Function.Injective pi ∧ normalPart (NonUnitalStarAlgHom.range
pi) = Set.univ) := by
```