The geometric structure family is the family of pure-state translate kernels

`MathlibAnnex.Analysis.CStarAlgebra.geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate`

theorem

Identifies geometric points with kernels of unitary translates of pure states.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. Let $S(A)$ be its original geometric structure space: its points are the sets obtained as unions of the kernels of all norm-one bounded linear functionals supporting maximal proper faces of the closed unit ball. Write $\mathcal{P}(A)$ for the pure states of A, that is, positive norm-one bounded complex-linear functionals $A \to \mathbb{C}$ that are extreme in that state space for real convexity.

For a unitary u in the forced unitization $\widetilde{A} = \mathbb{C} \oplus A$, define $(T_u\varphi)(a) = \varphi(ua)$. Then

$$S(A) = \{\ker(T_u\varphi) : \varphi \in \mathcal{P}(A),\ u \in U(\widetilde{A})\}. \qquad (1)$$

Here ua denotes the unitization action on the original ideal A; the functional and its kernel in this formula are on A.

Assumptions

The source requires the nonzero complex C*-algebra and its compatible order. Positivity and purity refer to functionals on A; the translating unitary is allowed to lie in its forced unitization. The kernel is complex-linear, even though the face and extreme-point conditions use real convexity. No positivity of the translated functional $T_u\varphi$ is required.

Conclusion

Every original geometric-structure point can be written using one pure state and one unitary, and every such pair gives a point. Formula (1) is a family equality, so two pairs with the same kernel give the same point. It does not assert uniqueness of a state–unitary pair or replace the original family by a new definition.

Main citations

- Exact pure-state translate family equality: `geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate`.

- Extreme-kernel family and pure-state unitary decomposition: the related family equality and translation identities for the same pure state and unitary.
- Ryotaro Tanaka, arXiv:2504.05551v1, Lemma 5.3: the paper's pure-state description of the geometric structure space.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate :
    NormedSpace.geometricStructureFamily ℂ A =
      {I : Set A | ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
        ∃ u : unitary (Unitization ℂ A), ((unitaryTranslate u phi).ker : Set A) = I}
```

In the source	Mathematical meaning
Exact declaration `{A : Type*}`, the nonunital C*-algebra, order and nontriviality instances	The common algebra and hypotheses of the two family descriptions.
Exact declaration `NormedSpace.geometricStructureFamily ℂ A`	The family $S(A)$ of geometric-structure points, each a subset of A obtained from a maximal face and its supporting functionals.
Exact declaration `{I : Set A | ...}`	A family described by a property of the entire subset $I \subseteq A$.
Exact declaration `∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧ ...`	A pure state $\varphi \in \mathcal{P}(A)$, with positivity, norm one and extremality as specified above.
Exact declaration `∃ u : unitary (Unitization ℂ A)`	One unitary of the forced unitization, for that same state. The state is the same one already quantified; the unitary is not required to belong to the original algebra.
Exact declaration `((unitaryTranslate u phi).ker : Set A) = I`	The equality $\ker(T_u\varphi) = I$, with the translated functional defined on A.
The complete equality **Exact declaration**	Equality of families of subsets: every left member has the displayed witnesses, and every such pair gives a left member. No unique parametrization is asserted.

Exact surrounding source context (separate excerpts)

Source lines 25–25:

```
variable {A : Type*} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Source lines 26–26:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact declaration for the source guide

```
theorem geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate :
    NormedSpace.geometricStructureFamily ℂ A =
      {I : Set A | ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
        ∃ u : unitary (Unitization ℂ A), ((unitaryTranslate u phi).ker : Set A) = I} := by
```