

A character separates a closed ideal

MathlibAnnex.Analysis.CStarAlgebra.exists_character_annihilating_closedIdeal_of_not_mem

Separates an element from a closed ideal of a commutative C*-algebra by a character vanishing on the ideal.

Statement

Let D be a commutative unital complex C*-algebra, J an ideal whose carrier is norm-closed, and $d \in D$ with $d \notin J$. There is a character χ of D that vanishes on every $x \in J$ while $\chi(d) \neq 0$. Properness follows from $d \notin J$ and is not separately assumed.

Assumptions

Let D be a commutative unital complex C*-algebra, J an ideal whose carrier is norm-closed, and $d \in D$ with $d \notin J$.

Conclusion

There is a character χ of D that vanishes on every $x \in J$ while $\chi(d) \neq 0$. Properness follows from $d \notin J$ and is not separately assumed.

Proof idea

The source transports J through the Gelfand star transform to an ideal of continuous functions, uses isometry to preserve closedness and $d \notin J$, then separates the transformed point by a character and pulls the annihilation/nonvanishing equations back.

Proof steps

1. Exact Lean statement: $\forall \{D : \text{Type } u\} [\text{inst} : \text{CommCStarAlgebra } D] (J : \text{Ideal } D), \text{IsClosed } \uparrow J \rightarrow \forall \{d : D\}, d \notin J \rightarrow \exists \chi, (\forall x \in J, \chi x = 0) \wedge \chi d \neq 0$
2. $\exists \chi, (\forall x : D, x \in J \rightarrow \chi x = 0) \wedge \chi d \neq 0$.
3. The source transports J through the Gelfand star transform to an ideal of continuous functions, uses isometry to preserve closedness and $d \notin J$, then separates the transformed point by a character and pulls the annihilation/nonvanishing equations back.

Main citations

Lean source signature (exact)

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theorem exists_character_annihilating_closedIdeal_of_not_mem
{D : Type u} [CommCStarAlgebra D]
(J : Ideal D) (hJclosed : IsClosed (J : Set D))
{d : D} (hd : d ∉ J) :
∃ chi : WeakDual.characterSpace ℂ D,
(∀ x : D, x ∈ J → chi x = 0) ∧ chi d ≠ 0
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[Read exact MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: 1e09b496fe0e06eeb046e141957245bfb51a6b1b7b8c28cf44ffdefc9574de2f

Card SHA-256: 6478c2932ffa998eda8ac724e6df08a7c18971e943cf48ac511aa19e7793261a

Approved exposition SHA-256: ec0a6da44dce8322e006bd4e79b375e404e1fe02c3846aa95e2db21df116b753

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