Realizing a shell family by a faithful irreducible operator algebra

`MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicShellModel`

theorem

Constructs unitary links between pure-state summands and an irreducible algebra containing a faithful copy of CAR.

Statement

Let A be the completed CAR algebra and fix a shell family as defined below. Let $\mathcal{H} = \bigoplus_{j \in I} H_j$ be the Hilbert direct sum of the chosen pure-state Gelfand-Naimark-Segal (GNS) spaces, $\pi = \bigoplus_j \pi_j$ their representation, and $b_j = e_j \xi_j$ the embedded unit cyclic vectors, where $\xi_j \in H_j$ is the selected unit cyclic vector and $e_j : H_j \to \mathcal{H}$ is coordinate inclusion. There exist unitaries L_j on $\mathcal{H}$ with $L_j b_j = b_o$, $L_j \pi(\alpha_j(p_n)) = \pi(w_{j,n})$ for every j, n, and $L_o = I_{\mathcal{H}}$. The norm-closed unital star algebra $T = C^*(\pi(A), \{L_j\})$ in the bounded operators $B(\mathcal{H})$ contains an injective copy of A and acts irreducibly on $\mathcal{H}$.

Assumptions

Let A be the completed CAR algebra with matrix stages $M_{2^n}(\mathbb{C})$ embedded by $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit, $p_n = f_n - f_{n+1}$, and ω for the root state characterized by $\omega(\iota_n(a)) = a_{00}$ on each stage, where ι_n is its canonical embedding.

Index chosen pure-state Gelfand-Naimark-Segal (GNS) representations by their unitary-equivalence classes $j \in I$, with root index o and representative states φ_j.

A fixed shell family consists of complex-linear unital star automorphisms α_j and elements $w_{j,n} \in A$ satisfying $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$.

At the root, α_o is the identity and $w_{o,n} = p_n$. The Hilbert spaces are the chosen complete complex GNS spaces. No countability of I is assumed. Irreducibility means a nonzero representation with no proper nonzero closed reducing subspace.

Conclusion

The shell identities are realized by actual bounded unitary operators on one Hilbert space. The generated algebra acts irreducibly while retaining the original algebra faithfully; rank-one limiting projections and irreducibility are proved consequences, not fields imposed on the input family.

Proof route

By the atomic common-range theorem, the decreasing projections $\pi(\alpha_j(f_n))$ start at the identity and have common range $\mathbb{C} b_j$; the root flag has common range $\mathbb{C} b_o$. The represented links $\pi(w_{j,n})$ identify their corresponding orthogonal difference shells. The cited atomic-shell construction joins these partial isometries and the map between the one-dimensional residual spaces into unitaries. Its irreducibility argument uses the inequivalent irreducible GNS summands and the links between their cyclic vectors. Faithfulness comes from the faithful root summand of π.

Proof steps

1. The support equations give initial projection $\pi(\alpha_j(f_n) - \alpha_j(f_{n+1}))$ and final projection $\pi(f_n - f_{n+1})$ for each represented shell link.

2. Use the proved common-range identity to identify the residual projections with the rank-one projections onto b_j and b_o. Apply the atomic-shell existence theorem, which supplies the unitaries and irreducibility of the generated algebra.

3. At the root the constructed link is the identity on every difference shell and on the residual line. These subspaces reconstruct $\mathcal{H}$, so $L_o = I_{\mathcal{H}}$. The faithful root summand makes π, hence its codomain restriction to T, injective.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellData · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedAtomicRepresentation · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedRootRepresentation_injective · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedAtomicRepresentation_injective · Exact source
- MathlibAnnex.CStarAlgebra.CAR.iInf_range_atomic_transportedFlag_eq_span · Exact source
- MathlibAnnex.CStarAlgebra.CAR.representativeLink_initial · Exact source
- MathlibAnnex.CStarAlgebra.CAR.representativeLink_final · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.exists_irreducible_pureAtomicShellModel · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isUniqueIrreducibleModel_shellFamilyInclusion · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem exists_completedAtomicShellModel (family : RepresentativeShellFamily) :
    ∃ L : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit →
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
          MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState,
      (∀ i, L i ∈ unitary
        (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
          MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState)) ∧
      (∀ i, L i (MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState i
          (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState i)) =
        MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState
          completedRootPureState.classOf
          (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState
            completedRootPureState.classOf)) ∧
      (∀ i n, (L i).comp (selectedAtomicRepresentation
          (transportedFlag family i n - transportedFlag family i (n + 1))) =
        representedShellLink family i n) ∧
      L completedRootPureState.classOf = 1 ∧
      Function.Injective
        (MathlibAnnex.CStarAlgebra.AtomicConstruction.sourceHom selectedAtomicRepresentation L) ∧
      Representation.IsIrreducible
        (MathlibAnnex.CStarAlgebra.AtomicConstruction.ambientInclusion selectedAtomicRepresentation L)
```

> Here `Limit` is A and `completedRootPureState` is the root state ω, bundled with its purity proof. `RepresentativeShellFamily` is the supplied family $(\alpha_j, w_{j,n})$, `SelectedAtomicHilbert completedRootPureState`

> is $\mathcal{H}$, and `selectedAtomicRepresentation` is π. The existentially quantified `L` gives the unitaries L_j: the conjunction records their action on the cyclic vectors and shells, the identity root link, source injectivity and irreducibility of the generated target. It does not assert existence of the input family.

Lean realization notes

This is existence of the operator links for a supplied family. It does not construct that family without hypotheses, nor prove uniqueness of every irreducible representation of T. The universal capture theorem supplies that additional conclusion. No generic homogeneity principle is a further hypothesis of this exact theorem.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:1ee4a51c5a889e36f8bce5ebb44b8bdb3f459aa589225113d4e8aa518643807f
Card revision: 1 · SHA-256: ee9b781557606ba28fdda63633cfb3efeb167a9f8fbebfe6a215537bd661711d
Exposition revision: 1 · SHA-256: e7453e7f2b99e61e071902e0d57d9689b7865fdf71f8c305b54ea518732264e5
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
Header SHA-256: 5d5ce65469648d7f2076676f11dc16b8e64d9cb1c3a1b1cfc91d8285d7cfae9c