The hull–kernel closure construction

`ClosureSpace.HullKernel.space`

def

The hull–kernel closure construction

Statement

A family of proper subsets of a set carries a natural closure structure. To close a subfamily, intersect its members and then take every member of the original family containing that intersection. The points of the resulting closure space are the subsets themselves.

Definition

Let X be a set and let $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfy $X \notin \mathcal{F}$. For $\mathscr{A} \subseteq \mathcal{F}$ and $a \subseteq X$, define the kernel and hull by

$$\begin{aligned} k(\mathscr{A}) &= \bigcap_{I \in \mathscr{A}} I, \\ h(a) &= \{I \in \mathcal{F} : a \subseteq I\}. \end{aligned}$$

The empty intersection is X. The **hull–kernel closure** on $\mathcal{F}$ is

$$c_{\mathcal{F}}(\mathscr{A}) = h(k(\mathscr{A})) = \{I \in \mathcal{F} : k(\mathscr{A}) \subseteq I\}.$$

There are two levels of subsets: a point I is a subset of X, while $\mathscr{A}$ is a set of such points. Accordingly, $k(\mathscr{A}) \subseteq X$ and $c_{\mathcal{F}}(\mathscr{A}) \subseteq \mathcal{F}$.

Both k and h reverse inclusion, so $c_{\mathcal{F}}$ is monotone. Every $I \in \mathscr{A}$ contains $k(\mathscr{A})$, giving extensivity. The cited kernel identity proves idempotence:

$$\begin{aligned} & k(h(k(\mathscr{A}))) = k(\mathscr{A}) \\ \Longrightarrow \quad & c_{\mathcal{F}}(c_{\mathcal{F}}(\mathscr{A})) = c_{\mathcal{F}}(\mathscr{A}). \end{aligned}$$

Finally,

$$c_{\mathcal{F}}(\varnothing) = h(X) = \{I \in \mathcal{F} : X \subseteq I\} = \varnothing,$$

because $X \notin \mathcal{F}$. These are the four closure-space axioms.

Assumptions

The only restriction on $\mathcal{F}$ is $X \notin \mathcal{F}$; the family itself may be empty. Its members are arbitrary proper subsets of X. No vector-space structure, topology, or closure of $\mathcal{F}$ under intersections is assumed.

Conclusion

The construction yields the closure space $(\mathcal{F}, c_{\mathcal{F}})$ with the displayed formula. A subfamily $\mathscr{A}$ is closed precisely when $h(k(\mathscr{A})) = \mathscr{A}$.

The separate separation theorem shows that this closure is T_0: distinct points have distinct singleton closures. If $\mathcal{F}$ is an antichain under inclusion, every singleton is closed, the T_1 property. The antichain condition belongs to that further theorem, not to the construction.

Main citations

- Kernel, hull and the idempotence identity: `ClosureSpace.HullKernel.kernel`, `ClosureSpace.HullKernel.hull` and `ClosureSpace.HullKernel.kernel_hull_kernel`, lines 30–66.
- Exact construction and closure formula: `ClosureSpace.HullKernel.space`, lines 69–89.
- The separation properties: `ClosureSpace.HullKernel.isT0_space` and `ClosureSpace.HullKernel.isT1_space_of_antichain`, lines 104–134.
- The four closure axioms: `ClosureSpace`, lines 28–30, with its laws at lines 44–61.
- arXiv:2504.05551v1, Definition 4.1: the later hull–kernel construction on geometric points. The generic construction here has purely set-theoretic assumptions.

Lean source signature (exact)

The carrier `X` is declared in the separate source context below.

```
def space (F : Set (Set X)) (hF : Set.univ ∉ F) : ClosureSpace F where
  toClosureOperator := ClosureOperator.mk'
    (fun s => hull F (kernel F s))
    (by
      intro s t hst
      exact hull_antitone F (kernel_antitone F hst))
    (subset_hull_kernel F)
    (by
      intro s
      rw [kernel_hull_kernel])
  map_empty' := by
    change hull F (kernel F ∅) = ∅
    rw [kernel_empty]
    apply Set.eq_empty_iff_forall_notMem.2
    intro I hI
    have hval : I.val = Set.univ := Set.Subset.antisymm (Set.subset_univ _) hI
    exact hF (hval ▸ I.property)
```

In the source	Mathematical meaning
Exact declaration `X, F : Set (Set X)`	An arbitrary set X and a family $\mathcal{F} \subseteq \mathcal{P}(X)$. A point $I \in \mathcal{F}$ is itself a subset of X (`I.val`).
Exact declaration `hF : Set.univ ∉ F`	$X \notin \mathcal{F}$: every member is proper. The family itself may be empty.
Exact declaration `kernel F s, hull F`	For `s` corresponding to $\mathscr{A} \subseteq \mathcal{F}$, $k(\mathscr{A}) = \bigcap_{I \in \mathscr{A}} I$, with empty intersection X, and $h(B) = \{I \in \mathcal{F} : B \subseteq I\}$.
Exact declaration `fun s => hull F (kernel F s)`	The whole closure operation $c_{\mathcal{F}}(\mathscr{A}) = \{I \in \mathcal{F} : \bigcap_{J \in \mathscr{A}} J \subseteq I\}$ on subfamilies of $\mathcal{F}$.
Exact declaration `ClosureOperator.mk', map_empty'`	The construction certifies monotonicity, extensivity and idempotence of this operation, and $c_{\mathcal{F}}(\varnothing) = \varnothing$ using `hF`.
Exact declaration `: ClosureSpace F`	The returned closure-space structure is on the point set $\mathcal{F}$, with all four closure axioms; a topology on X or finite-union preservation is not part of the output.

Source context (separate exact excerpt)

The surrounding declarations below are from line 27; their mathematical meaning is included in the guide above.

```
variable {X : Type u}
```

Exact declaration for the source guide

```
def space (F : Set (Set X)) (hF : Set.univ ∉ F) : ClosureSpace F where
  toClosureOperator := ClosureOperator.mk'
    (fun s => hull F (kernel F s))
    (by
      intro s t hst
      exact hull_antitone F (kernel_antitone F hst))
    (subset_hull_kernel F)
    (by
      intro s
      rw [kernel_hull_kernel])
  map_empty' := by
    change hull F (kernel F ∅) = ∅
    rw [kernel_empty]
    apply Set.eq_empty_iff_forall_notMem.2
    intro I hI
    have hval : I.val = Set.univ := Set.Subset.antisymm (Set.subset_univ _) hI
    exact hF (hval ▸ I.property)

@[simp]
```