

Pure state detecting a nonzero square

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_nonzero_on_star_mul_self`

Supplies a pure state that detects $a^* a$, enabling faithfulness arguments without separability of the algebra.

Statement

Let A be a nonzero unital complex C^* -algebra and let $a \in A$ be nonzero. No separability assumption on A is required. Some pure state ϕ in $\text{stateSpace } A$ satisfies $\phi(a^* a) \neq 0$.

Assumptions

Let A be a nonzero unital complex C^* -algebra and let $a \in A$ be nonzero. No separability assumption on A is required.

Conclusion

Some pure state ϕ in $\text{stateSpace } A$ satisfies $\phi(a^* a) \neq 0$.

Proof idea

The source sets $b = a^* a$, uses its nonzero spectral radius to obtain a nonzero character value on the closed elemental commutative star subalgebra generated by b , extends that character to a pure state, and transfers the nonzero value back to b .

Proof steps

1. Exact Lean statement: $\forall \{A : \text{Type } u\} [\text{inst} : \text{CStarAlgebra } A] [\text{inst}_1 : \text{PartialOrder } A] [\text{inst}_2 : \text{StarOrderedRing } A] [\text{Nontrivial } A] \{a : A\}, a \neq 0 \rightarrow \exists \phi \in \text{MathlibAnnex.Analysis.CStarAlgebra.stateSpace } A, \text{MathlibAnnex.Analysis.CStarAlgebra.IsPureState } A \phi \wedge \phi(\text{star } a * a) \neq 0$
2. $a \neq 0 \rightarrow \exists \phi \in \text{stateSpace } A, \text{IsPureState } A \phi \wedge \phi(\text{star } a * a) \neq 0$.
3. The source sets $b = a^* a$, uses its nonzero spectral radius to obtain a nonzero character value on the closed elemental commutative star subalgebra generated by b , extends that character to a pure state, and transfers the nonzero value back to b .

Main citations

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_extension`

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem exists_pureState_nonzero_on_star_mul_self [Nontrivial A]
{a : A} (ha : a ≠ 0) :
  ∃ phi : A →L[ℂ] ℂ,
  phi ∈ stateSpace A ∧ IsPureState A phi ∧ phi (star a * a) ≠ 0
```

[Read exact MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: 23a70d1e81d7c3d1ed7932a2de55be2e240d36d087130ed0dd8e21ce1e3d7013

Card SHA-256: 80afe23433c02265252a97ff93dd683f25f5d0dbab62a98400823808de83329a

Approved exposition SHA-256: 10e830eb3003db902b8084311bbed7c1cc308d086d041fa875134d5de6ee7ade

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source-exposition correspondence, not an independent proof audit.