

Faithfulness of the representation on the CAR trace Hilbert space

MathlibAnnex.CStarAlgebra.CAR.traceModelRepresentation_injective

theorem

Unitary conjugation preserves injectivity of the target-state representation.

Statement

Let A be the completed CAR algebra and τ its normalized trace. Fix a representative shell family and its chosen unitary links L_j in the selected atomic representation π . Write $T = C^*(\pi(A), \{L_j\})$ for the resulting concrete target and $\iota : A \rightarrow T$, $\iota(a) = \pi(a)$, for its faithful unital source map. This fixes one target and one source map throughout. Let $(H_\phi, \rho_\phi, \xi_\phi)$ be the GNS representation of the chosen state extension ϕ , and let $(H_\tau, \pi_\tau, \xi_\tau)$ be the CAR trace GNS representation. Choose once the pointed unitary $E : H_\tau \rightarrow H_\phi$ with $E\xi_\tau = \xi_\phi$ and $E\pi_\tau(a) = \rho_\phi(\iota(a))E$. Then $\tilde{\rho}(t) = E^{-1}\rho_\phi(t)E$ is injective on T .

Assumptions

The family, extension, and chosen unitary are the ones used to define `traceModelRepresentation`. Faithfulness of ρ_ϕ is the separately proved kernel-dichotomy result.

Conclusion

The same concrete target admits a faithful representation on the nonzero separable space $H_\tau = L^2(A, \tau)$. Its restriction to A is the CAR trace representation.

Proof route

Undo the same conjugation and apply faithfulness of the original target-state representation.

Proof steps

1. Suppose $\tilde{\rho}(t) = \tilde{\rho}(s)$. Since E is a unitary onto H_ϕ ,

$$\rho_\phi(t) = E\tilde{\rho}(t)E^{-1} = E\tilde{\rho}(s)E^{-1} = \rho_\phi(s).$$

The theorem `tracialRepresentation_injective` then gives $t = s$. In the source this is the composition of the injective conjugation equivalence with the injective representation; it does not require another ideal argument.

Main citations

- The stated existence or structural result · Exact source
- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- Definition and direction of the transport · Exact source
- Faithfulness before conjugation · Exact source

- Separable CAR trace space · Exact source
- Source action in the transported model · Exact source

Lean source signature (exact)

```
theorem traceModelRepresentation_injective (family : RepresentativeShellFamily) :
  Function.Injective (traceModelRepresentation family)
```

`traceModelRepresentation family` is $\tilde{\rho}$ on `TraceHilbertSpace`. The proof uses `(traceGNSUnitary family).symm.conjStarAlgEquiv.injective` and `tracialRepresentation_injective family`. These refer to the exact same E and ρ_ϕ as the definition, not newly selected existence witnesses.

Lean realization notes

Injectivity concerns the homomorphism into operators, not a vector functional. The result does not assert irreducibility of this representation or norm separability of T . Conjugation uses the inverse on the left because E points from the CAR space to the target-state space.

▼ Exact Card identity

Language: en

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