A finite-dimensional compact model with a dimension condition

`ClosureSpace.HasFiniteDimensionalCompactHomeomorph`

def

Requires a finite-dimensional compact model satisfying a specified condition on its dimension.

Statement

Let (X, c) be a closure space and let $P(n)$ be a condition on a natural number n. The finite-dimensional compact-homeomorphism condition holds when there exist a nonzero finite-dimensional complex Hilbert space H satisfying $P(\dim_{\mathbb{C}} H)$ and a full homeomorphism $(X, c) \to S(\mathcal{K}(H))$.

Definition

Here $\mathcal{K}(H)$ denotes the algebra of compact operators on H. For a complex normed space D, $S(D)$ consists of the unions of kernels of norm-one bounded complex-linear functionals supporting each maximal proper face of its closed unit ball. Faces use real convexity. The support-functional norm theorem justifies the norm-one description for these nonempty faces. Its closure is $c_D(\mathcal{U}) = \{I \in S(D) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$.

All the following data refer to the same H: its norm, complex inner product and completeness, nontriviality, finite-dimensionality, the truth of $P(\dim_{\mathbb{C}} H)$, and a full closure-space homeomorphism

$$e : (X, c) \longrightarrow S(\mathcal{K}(H)).$$

Full continuity applies to arbitrary subsets in both directions. Finite-dimensionality is required separately from the condition on its finite complex dimension.

Assumptions

The input consists only of c and P. The condition requires the existence of the Hilbert space, its finite-dimensionality, and the stated dimension condition. Nonzero H is explicit in the literal right-hand side. No separability hypothesis is added. Here the natural number dimension is finite complex vector-space dimension; later infinite-dimensional comparisons use Hilbert-basis cardinality separately.

Conclusion

One finite-dimensional Hilbert space must satisfy both the dimension condition and the homeomorphism requirement. Taking $P(m)$ to be $m = n$ specifies an exact dimension; taking it to be $m \leq n$ specifies an upper bound. Neither choice replaces the explicit finite-dimensionality and

nonzero-model conditions.

Main citations

- Exact definition.
- Finite-comparison counterpart.

Lean source signature (exact)

Exact source and declaration. The complete defining right-hand side is included.

```
def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧
            Nonempty (Homeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

In the source	Mathematical meaning

Input closure and dimension condition
`def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}`
`(c : ClosureSpace X) (P : ℕ → Prop) : Prop :=`
	The input c is the closure on X. The second input is the condition $P(n)$ on a natural number n. The following existential statement defines the requested model condition; it does not select a preferred model or map.

One normed model space
`∃ (H : Type uH) (instN : NormedAddCommGroup H),`
`letI := instN`
	There must exist one normed additive group H. The first `letI` installs that chosen normed structure on H for the inner-product type that follows.

Complex inner product on that model
`∃ instI : InnerProductSpace ℂ H,`
`letI := instI`
	The same H must have a compatible complex inner product. This `letI` installs it for the later Hilbert-space and operator-algebra types.

Completeness of that model
`∃ instC : CompleteSpace H,`
`letI := instC`
	The same normed space must be complete. The next `letI` installs this completeness, used in the compact-operator algebra on this Hilbert space.

A nonzero model
`∃ instT : Nontrivial H,`
`letI := instT`
	The same H must be nonzero. Its `letI` makes this nontriviality available in the remaining conditions; it does not introduce another space.

Finite-dimensionality and its dimension condition `FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧`	The same H must be finite-dimensional, and its finite complex dimension must satisfy $P(\dim_{\mathbb{C}} H)$. These are two conditions. The finite-dimensionality requirement is not replaced by a condition on `finrank`.

In the source	Mathematical meaning
Existence of a homeomorphism to the same model `Nonempty (Homeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`	`Nonempty` requires an element of the displayed homeomorphism type: one bijection $e : X \to S(\mathcal{K}(H))$ with continuity in both directions. The complete target argument is the geometric closure $c_{\mathcal{K}(H)}$ on the compact-operator algebra of the same H used in all preceding conditions.
The homeomorphism and its inverse `structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun continuous_invFun : d.Continuous c invFun`	The inherited equivalence gives e and its inverse e^{-1}. For every $E \subseteq X$ and $F \subseteq S(\mathcal{K}(H))$, the two fields require, respectively, $e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E))$, $e^{-1}(c_{\mathcal{K}(H)}(F)) \subseteq c(e^{-1}(F))$. Both use the same model and the same map; the definition does not choose separate forward and inverse maps.
Continuity for all subsets `def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop := ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`	For $r : (Y, d) \to (Z, k)$, the definition requires $r(d(E)) \subseteq k(r(E)) \quad (E \subseteq Y)$. There is no finiteness restriction in this condition.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
namespace ClosureSpace

attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
```

The model-space level `uH` and the input-space level `uX` are independent fixed parameters. All model structures and conditions above refer to the same quantified H.

Related exact source for the reading guide

Continuity for all subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The homeomorphism and its inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```