Forward continuity of normal parts preserves Hilbert-basis cardinality

`MathlibAnnex.Analysis.CStarAlgebra.hilbertBasis_lift_mk_eq_of_normalPart_continuous`

theorem

Preserves Hilbert-basis cardinality under forward full continuity of a finite normal-part homeomorphism.

Statement

Let H, K be complex Hilbert spaces. Let $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. No unit is required. The inner product is linear in its second variable. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define the bounded linear functionals $f^{A_1}_{x,y} : A_1 \to \mathbb{C}$ and $f^{A_2}_{z,w} : A_2 \to \mathbb{C}$ by $f^{A_1}_{x,y}(a) = \langle y, a(x)\rangle$ and $f^{A_2}_{z,w}(b) = \langle w, b(z)\rangle$. Their kernels form the normal parts

$$N(A_1) = \{\ker f^{A_1}_{x,y} : x, y \in H \setminus \{0\}\}, \qquad N(A_2) = \{\ker f^{A_2}_{z,w} : z, w \in K \setminus \{0\}\}. \quad (1)$$

Write c_{A_1} and c_{A_2} for the relative geometric closures on these normal parts. Thus, for $\mathcal{D} \subseteq N(A_1)$, $c_{A_1}(\mathcal{D}) = \{I \in N(A_1) : \bigcap_{J \in \mathcal{D}} J \subseteq I\}$, and similarly for A_2. Let $e : N(A_1) \to N(A_2)$ be a bijection satisfying $e(c_{A_1}(\mathcal{D})) = c_{A_2}(e(\mathcal{D}))$ for every finite $\mathcal{D} \subseteq N(A_1)$. This is the finite-homeomorphism condition. Assume in addition that its forward map is fully continuous:

$$e(c_{A_1}(\mathcal{D})) \subseteq c_{A_2}(e(\mathcal{D})) \qquad (\mathcal{D} \subseteq N(A_1)). \qquad (2)$$

For any Hilbert bases $b = (b_i)_{i \in \iota}$ of H and $c = (c_j)_{j \in \kappa}$ of K, their index cardinalities satisfy

$$|\iota| = |\kappa|. \qquad (3)$$

The invariant is Hilbert-basis index cardinality, not the cardinality of the vector spaces as sets and not their Hamel dimension.

Assumptions

The bases are arbitrary complete orthonormal families. They need not be countable, and neither space is assumed separable. No infinite-dimensionality, same-type condition on two projective factors, or full continuity of e^{-1} is assumed. Dimensions one and two are included through the finite-dimensional branch. The inclusion representations are nonzero and irreducible, and the algebras are norm closed; unitality is not added.

Conclusion

Forward full continuity preserves Hilbert-basis index cardinality, without assuming full continuity of the inverse. The conclusion neither chooses a basis-to-basis map nor asserts an isometry.

Main citations

The normal-part definition and coefficient-kernel membership theorem identify (1) under the stated closedness and irreducibility assumptions.

The Hilbert-dimension theorem gives (3) from forward full continuity. The related results are the finite-dimensional comparison, the continuous first lift and its inverse, and Hilbert-basis comparison.

Lean source signature (exact)

The preceding `variable` declarations introduce the two Hilbert spaces, the closed algebras, their irreducible actions, and `e`. The `include hS hT in` at the effective inclusion retains the irreducibility assumptions for this theorem. The map `e` is also a parameter because it occurs in `hcont`; `e.toEquiv` is the same bijection with its finite-continuity fields omitted. The following contiguous prefix ends before the proof assignment.

```
theorem hilbertBasis_lift_mk_eq_of_normalPart_continuous
    (hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous
      ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)
    (b : HilbertBasis ι ℂ H) (c : HilbertBasis κ ℂ K) :
    Cardinal.lift.{uJ} (#ι) = Cardinal.lift.{uI} (#κ)
```

In the source, `S` and `T` denote the algebras A_1 and A_2 used here, respectively.

In the source	Mathematical meaning

Two Hilbert spaces and basis indices
`universe u v uI uJ`
`variable {H : Type u} [NormedAddCommGroup H]`
`[InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K]`
`[InnerProductSpace ℂ K] [CompleteSpace K]`
`variable {ι : Type uI} {κ : Type uJ}`
	The spaces H, K are complex Hilbert spaces, including completeness. The sets ι, κ index their Hilbert bases.

In the source	Mathematical meaning

Closed irreducible operator algebras
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The algebras A_1, A_2 are norm-closed complex $*$-subalgebras of $\mathcal{B}(H), \mathcal{B}(K)$. Their inclusion representations are nonzero and irreducible. No unit is required.

The same finite comparison of normal parts
`variable (e : ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))`
	The relative closures c_{A_1}, c_{A_2} are those on $N(A_1), N(A_2)$, rather than on the whole geometric structure spaces. One bijection $e : N(A_1) \to N(A_2)$ satisfies, for every finite $\mathcal{D} \subseteq N(A_1)$,
	$$e(c_{A_1}(\mathcal{D})) = c_{A_2}(e(\mathcal{D})).$$

Effective irreducibility assumptions `include hS hT in`	The preceding inclusion retains both irreducibility assumptions as theorem parameters. The already declared e is also a parameter because it occurs in the following continuity assumption.

Forward continuity for all subsets
`(hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)`
	The same bijection e is fully continuous in the forward direction: for every $\mathcal{D} \subseteq N(A_1)$,
	$$e(c_{A_1}(\mathcal{D})) \subseteq c_{A_2}(e(\mathcal{D})).$$
	This is (2); full continuity of e^{-1} is not an input.

Arbitrary Hilbert bases `(b : HilbertBasis ι ℂ H) (c : HilbertBasis κ ℂ K) :`	The bases are $b = (b_i)_{i \in \iota}$ of H and $c = (c_j)_{j \in \kappa}$ of K: arbitrary complete orthonormal families.
Equality of basis-index cardinalities `Cardinal.lift.{uJ} (#ι) = Cardinal.lift.{uI} (#κ) := by`	The two basis-index sets have the same cardinality: $$\lvert\iota\rvert = \lvert\kappa\rvert,$$ as in (3). The lifts allow Lean to compare cardinalities belonging to different universes. This is Hilbert-basis cardinality, not Hamel dimension or the cardinality of the vector spaces as sets.

In the source	Mathematical meaning

Finite continuity of the map and its inverse
`structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  finitelyContinuous_toFun : c.FinitelyContinuous d toFun`
`  finitelyContinuous_invFun : d.FinitelyContinuous c invFun`
	The inherited equivalence gives e and e^{-1}. Its fields require finite continuity in both directions: for finite $\mathcal{D} \subseteq N(A_1)$ and finite $\mathcal{E} \subseteq N(A_2)$, respectively,
	$$e(c_{A_1}(\mathcal{D})) \subseteq c_{A_2}(e(\mathcal{D})),$$
	$$e^{-1}(c_{A_2}(\mathcal{E})) \subseteq c_{A_1}(e^{-1}(\mathcal{E})).$$
	This is the finite-homeomorphism condition used in the Statement.

Exact surrounding source context (separate excerpts)

NormalPartHilbertDimension.lean:

```
universe u v uI uJ
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable {ι : Type uI} {κ : Type uJ}
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.FiniteHomeomorph
  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))
```

NormalPartHilbertDimension.lean:

```
include hS hT in
```

Related exact source for the reading guide

Equality of basis-index cardinalities · Fixed source

```
    Cardinal.lift.{uJ} (#ι) = Cardinal.lift.{uI} (#κ) := by
```

Finite continuity of the map and its inverse · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```