A compact-operator model for a representation

`MathlibAnnex.Analysis.CStarAlgebra.IsCompactOperatorModel`

def

Packages faithfulness together with equality of the represented image and the compact operators.

Statement

Let A be a complex C^*-algebra, with no unit assumed, let H be a complex Hilbert space, and let $e : A \to \mathcal{B}(H)$ be a $*$-representation. The predicate below asserts that e identifies A exactly with the compact operators on H.

Definition

The three conditions are

$$e \text{ is injective}, \qquad e(A) \subseteq \mathcal{K}(H), \qquad \mathcal{K}(H) \subseteq e(A).$$

Equivalently, e is faithful and $e(A) = \mathcal{K}(H)$; hence A is $*$-isomorphic to $\mathcal{K}(H)$ through the specified representation.

Assumptions

No separability, irreducibility or singleton-spectrum hypothesis is part of this definition. A unit is not assumed.

Conclusion

The output is a property of the given representation e, not merely an assertion that some abstract $*$-isomorphism exists.

Main citations

- Exact definition and context

Lean source signature (exact)

The complete definition is copied from the pinned source.

```
/-- A nonunital star representation that is injective and has precisely the
compact operators as its range.  This is the unbundled form of a star
isomorphism with `K(H)`. -/
def IsCompactOperatorModel [NonUnitalCStarAlgebra A]
    (e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H)) : Prop :=
  Function.Injective e ∧
    (∀ a : A, IsCompactOperator (e a)) ∧
    ∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T
```

In the source	Mathematical meaning
Exact declaration `[NonUnitalCStarAlgebra A]`	The complex C^*-algebra A; a unit is not required.
Exact declaration `(e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H))`	The given $*$-representation $e : A \to \mathcal{B}(H)$.
Exact declaration `Function.Injective e`	Equal represented operators imply equal algebra elements.
Exact declaration `(∀ a : A, IsCompactOperator (e a))`	Every element a of A is sent to a compact operator; this is $e(A) \subseteq \mathcal{K}(H)$.
Exact declaration `∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T`	Choose any bounded operator T; if it is compact, choose an element a of the original algebra whose image is exactly T. This is the opposite inclusion.
Exact declaration `: Prop`	The conjunction of injectivity and the two inclusions $e(A) \subseteq \mathcal{K}(H)$ and $\mathcal{K}(H) \subseteq e(A)$.

Exact surrounding source context (separate excerpts)

```
universe u v

variable {A : Type u}
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```