

The decreasing root projections in the CAR completion

MathlibAnnex.CStarAlgebra.CAR.rootFlag

def

The distinguished stage corners give a nested sequence of projections supporting the root state.

Statement

Let A be the completed CAR algebra: the norm completion of the increasing union of $A_n = M_{2^n}(\mathbb{C})$ under the unital embeddings $a \mapsto a \otimes I_2$ (with the fixed coordinate reindexing). Write $\iota_n : A_n \rightarrow A$ for the canonical isometric inclusion and $e_{ij}^{(n)} = \iota_n(E_{ij})$ for its matrix units. Let ω be the product-vector state determined by $\omega(\iota_n(a)) = a_{00}$. Define $f_n = \iota_n(E_{00}) = e_{00}^{(n)}$. This sequence is the root flag.

Definition

At stage n , the matrix E_{00} has a single nonzero entry, equal to 1 at the distinguished diagonal position. Send it into A by ι_n . At the next stage, the new distinguished projection lies under the image of the preceding one, giving the decreasing relation. The finite matrix identity $E_{00}aE_{00} = a_{00}E_{00}$, together with the coordinate-preserving embeddings, explains the compression formula at later stages.

Assumptions

The zero coordinate is the same distinguished coordinate used by the standard stage inclusions. The index n ranges over all natural numbers.

Conclusion

Each f_n is a self-adjoint projection, $f_0 = 1$, $\omega(f_n) = 1$, and $f_{n+1} \leq f_n$. For an element from stage m , compression by f_n at any later stage $n \geq m$ is exactly scalar: $f_n \iota_m(a) f_n = a_{00} f_n$. The cited results establish these properties of the defined sequence.

Main citations

- Definition and its exact construction · Exact source
- Finite root projection · Exact source
- Finite root compression · Exact source
- Initial root projection · Exact source
- Projection property · Exact source
- State support · Exact source
- Decreasing root flag · Exact source
- Compression at later stages · Exact source

Lean source signature (exact)

```
noncomputable def rootFlag (n : ℕ) : Limit :=  
  ofStage n (rootProjection n)
```

Here `rootProjection n` is $E_{00} \in A_n$, `ofStage n` is ι_n , and `rootFlag n` is $f_n \in A$. The exact right-hand side uses these two operations only. The state ω in the prose is `rootState`; the projection, order, and compression facts have their own exact declarations.

Lean realization notes

The term rank one refers to E_{00} inside the finite matrix algebra. It does not assert that the image of f_n has one-dimensional range in a representation of the completed algebra. No norm or strong limit of the flag is part of this definition.

▼ Exact Card identity

Language: en

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