Points of the geometric structure space

`NormedSpace.GeometricStructureSpace`

def

Points of the geometric structure space

Statement

The geometric structure space of a normed space has as its points the support-kernel unions arising from maximal proper real convex faces of the closed unit ball. A point is the resulting subset of the vector space. Faces yielding the same subset represent the same point.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, with closed unit ball $B_E = \{x \in E : \|x\| \le 1\}$. Let $\mathfrak{M}_E$ be the family of maximal proper real convex faces of B_E. Explicitly, a member M is a nonempty convex subset of B_E, different from B_E, such that

$$\begin{aligned} x, y \in B_E, \quad 0 < t < 1, \quad (1-t)x + ty \in M \\ \implies \quad x, y \in M, \end{aligned}$$

and every proper convex face containing M equals M. The scalar t is real, including when $\mathbb{K}$ is complex.

Write E^* for the continuous $\mathbb{K}$-linear dual. For $M \in \mathfrak{M}_E$, define

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) &= \{f \in E^* : \|f\| \le 1,\ f|_M = 1\}, \\ I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f. \end{aligned}$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

The family of geometric points is

$$\mathcal{F}_{\mathbb{K}}(E) = \{I_{\mathbb{K}}(M) : M \in \mathfrak{M}_E\}.$$

The **geometric structure space** $S_{\mathbb{K}}(E)$ has precisely these subsets as its points. Thus an element $I \in S_{\mathbb{K}}(E)$ is a subset $I \subseteq E$ with a face M satisfying $I = I_{\mathbb{K}}(M)$. Choosing a representing face does not add data to the point. The assignment $M \mapsto I_{\mathbb{K}}(M)$ is not asserted to be injective.

Assumptions

The space E is normed over $\mathbb{K}$; completeness and nontriviality are not required. Convexity and the face condition are taken over the reals. Exposedness and supporting-functional existence are not additional conditions on the maximal-face predicate.

Conclusion

The resulting point set is $S_{\mathbb{K}}(E)$. Each point is a proper subset of E: if $I = I_{\mathbb{K}}(M)$ and $x \in M$, then $f(x) = 1$ for every $f \in \Sigma_{\mathbb{K}}(M)$, so $x \notin I$. Consequently $E \notin \mathcal{F}_{\mathbb{K}}(E)$, as required by the later hull–kernel closure construction.

The cited zero-space result gives $S_{\mathbb{K}}(E) = \varnothing$ when $E = \{0\}$. The present declaration specifies the points; the closure operation is a separate construction.

Main citations

- Maximal proper real convex faces: `NormedSpace.IsMaximalFace`, lines 32–39; its zero-space exclusion, lines 45–52.
- Supporting functionals and their kernel union: `NormedSpace.supportFunctionals`, lines 42–43; `NormedSpace.supportKernelUnion`, lines 108–109, and properness, lines 130–142.
- The family and the exact point type: `NormedSpace.geometricStructureFamily` and `NormedSpace.GeometricStructureSpace`, lines 37–54; the zero-space result, lines 73–86.
- arXiv:2504.05551v1, Definition 4.1: the face-based construction. The formal source separates its carrier from its closure and allows normed spaces without completeness or nontriviality.

Lean source signature (exact)

The following definition and the related family definition share the surrounding scalar and space context reproduced below.

```
def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$; it may be zero and need not be complete.
Exact declaration `I : Set E`	A subset $I \subseteq E$, regarded as a possible geometric point.

In the source	Mathematical meaning
Exact declaration `I ∈ geometricStructureFamily 𝕜 E`	The condition $I \in \mathcal{F}_{\mathbb{K}}(E)$: there exists a maximal proper real convex face M of the closed unit ball with $I_{\mathbb{K}}(M) = I$.
Exact declaration `{I : Set E // I ∈ geometricStructureFamily 𝕜 E}`	The complete defining type $S_{\mathbb{K}}(E)$ consists of such subsets with their membership condition. A point `p` has underlying subset `p.val`; no chosen face is stored.
Exact declaration `: Type u`	The output is the point set $S_{\mathbb{K}}(E)$, before any closure-space structure is attached.

Related definition — separate exact excerpt. The family used above is defined at lines 37–39 of the same source file.

```
def geometricStructureFamily : Set (Set E) :=
  letI : NormedSpace ℝ E := NormedSpace.restrictScalars ℝ 𝕜 E
  {I | ∃ F : Set E, IsMaximalFace F ∧ supportKernelUnion 𝕜 F = I}
```

In the source	Mathematical meaning
Exact declaration `letI ... := NormedSpace.restrictScalars ℝ 𝕜 E`	Convexity and faces use the real vector-space structure induced from the given $\mathbb{K}$-space; this is not an extra choice or assumption.
Exact declaration `∃ F : Set E, IsMaximalFace F ∧ supportKernelUnion 𝕜 F = I`	One subset M must satisfy both conditions: it is a maximal proper real convex face of the closed unit ball and its support-kernel union equals I. Here the source name `F` denotes M.
Exact declaration `: Set (Set E)`	The full RHS defines the family $\mathcal{F}_{\mathbb{K}}(E)$ of resulting subsets. Equal subsets give the same member even when their witnessing faces differ; no face is chosen as part of the output.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}

/-- The hull-kernel closure on `S(E)`. -/
```