# An irreducible operator algebra constructed from projection shells

`MathlibAnnex.CStarAlgebra.AtomicConstruction.exists_irreducible_atomicShellModel`

theorem

Shell partial isometries with rank-one limiting defects can be completed to unitary links that join inequivalent irreducible fibers.

## Statement

Let $A$ be a unital complex C*-algebra and $(\pi_i : A \to B(H_i))_{i \in I}$ a family of pairwise unitarily inequivalent irreducible unital representations on nonzero complete complex Hilbert spaces. Fix a root $o \in I$ and unit vectors $\xi_i \in H_i$. On $\mathcal{H} = \bigoplus_i H_i$, let $e_i$ be coordinate inclusion, $b_i = e_i \xi_i$, and $\pi = \bigoplus_i \pi_i$. For shell data satisfying the conditions below, there are unitaries $L_i \in B(\mathcal{H})$ such that $L_i b_i = b_o$ and $L_i(P_{i,n} - P_{i,n+1}) = W_{i,n}$ for all $i, n$. The literal inclusion into $B(\mathcal{H})$ of the norm-closed unital star algebra $T = C^*(\pi(A), \{L_i : i \in I\})$ is irreducible.

## Assumptions

For each $i \in I$, let $(U_{i,n})_{n \geq 0}$ and $(V_{i,n})_{n \geq 0}$ be decreasing subspaces of $\mathcal{H}$, with $U_{i,0} = V_{i,0} = \mathcal{H}$. All these subspaces and both intersections admit orthogonal projections. Write $P_{i,n} = P_{U_{i,n}}$, $Q_{i,n} = P_{V_{i,n}}$, and $P_{i,\infty} = P_{\bigcap_n U_{i,n}}$, $Q_{i,\infty} = P_{\bigcap_n V_{i,n}}$. The given bounded shell maps $W_{i,n}$ satisfy $W_{i,n}^* W_{i,n} = P_{i,n} - P_{i,n+1}$ and $W_{i,n} W_{i,n}^* = Q_{i,n} - Q_{i,n+1}$. Their limiting defects are prescribed: $P_{i,\infty} = \theta_{b_i,b_i}$ and $Q_{i,\infty} = \theta_{b_o,b_o}$. Here $\theta_{a,b} x = \langle b, x \rangle a$, with the inner product linear in its second variable. No finite, countable or separable restriction is placed on the family. The rank-one conditions are input hypotheses of this generic theorem.

## Conclusion

The same chosen family $(L_i)_i$ simultaneously has the unitary, vector-link and termwise-shell properties, and its concrete generated target acts irreducibly on $\mathcal{H}$. All closed reducing subspaces of that target action are therefore $0$ or $\mathcal{H}$.

## Proof route

For each $i$, sum the orthogonal shell maps strongly to an operator $S_i$, then fill its one-dimensional defect by $L_i = S_i + \theta_{b_o,b_i}$. To prove irreducibility, a bounded operator commuting with the target first has scalar diagonal blocks and zero off-diagonal blocks relative to the atomic source. Commuting with the links forces all those scalars to agree.

## Proof steps

1. Apply the cited projection-shell rank-one completion theorem separately to each $i$, with initial vector $b_i$ and final vector $b_o$. Coordinate inclusion is isometric, so both vectors have norm one. The decreasing normalized flags, the two support equations and the two limiting projection identities are exactly its hypotheses. It supplies $S_i$ with strong convergence $\sum_{n<N} W_{i,n} \to S_i$ and the unitary completion $L_i = S_i + \theta_{b_o,b_i}$. Choose these witnesses once for all $i$.

2. The completion theorem also gives $L_i b_i = b_o$ and $L_i(P_{i,n} - P_{i,n+1}) = W_{i,n}$. The rank-one correction vanishes on every initial difference shell because $b_i$ belongs to the limiting intersection. Thus the correction does not change any prescribed shell action.

3. Let $R \in B(\mathcal{H})$ commute with $T$. Since $\pi(A) \subseteq T$ and $L_i \in T$, it commutes with the atomic source and every link. The cited atomic-commutant theorem applies to the given irreducible, inequivalent fibers: each matrix block $e_j^* R e_i$ intertwines the corresponding actions; the Schur argument makes diagonal blocks scalar, while irreducibility and unitary inequivalence force off-diagonal blocks to vanish.

4. Write $Re_i x = e_i(z_i x)$. Commuting with $L_i$ and evaluating at $b_i$ gives $z_i b_o = z_o b_o$, hence $z_i = z_o$ since $\|b_o\| = 1$. Finite-support vectors are dense in the arbitrary-index Hilbert sum, so continuity yields $R = z_o 1$ on all of $\mathcal{H}$. Finally, the projection onto any closed reducing subspace commutes with $T$; being a scalar projection, it is $0$ or $1$. This is the cited scalar-commutant irreducibility criterion.

## Main citations

- The stated existence or structural result · Exact source
- Rank-one completion of each strongly summed shell family · Exact source
- Atomic commutant made scalar by the vector links · Exact source
- Scalar commutant implies irreducibility · Exact source
- Concrete generated target · Exact source
- Literal inclusion of the concrete target · Exact source

## Lean source signature (exact)

```
theorem exists_irreducible_atomicShellModel
    (pi : ∀ i, Representation A (H i))
    (hirr : ∀ i, StarAlgHom.IsIrreducible (pi i))
    (hno : ∀ ⦃i j : I⦄, i ≠ j → ∀ e : H i ≃ₗᵢ[ℂ] H j,
      ¬ StarAlgHom.Intertwines (pi i) (pi j) (e : H i →L[ℂ] H j))
    (o : I) (xi : ∀ i, H i) (hxi : ∀ i, ‖xi i‖ = 1)
    (W : I → ℕ → HilbertSum H →L[ℂ] HilbertSum H)
    (U V : I → ℕ → Submodule ℂ (HilbertSum H))
    [∀ i n, (U i n).HasOrthogonalProjection]
    [∀ i n, (V i n).HasOrthogonalProjection]
    [∀ i, (⨅ n, U i n).HasOrthogonalProjection]
    [∀ i, (⨅ n, V i n).HasOrthogonalProjection]
    (hU : ∀ i, Antitone (U i)) (hV : ∀ i, Antitone (V i))
    (hU0 : ∀ i, U i 0 = ⊤) (hV0 : ∀ i, V i 0 = ⊤)
    (hInitial : ∀ i n, ((W i n)†).comp (W i n) =
      Submodule.projectionShell (U i) n)
    (hFinal : ∀ i n, (W i n).comp ((W i n)†) =
      Submodule.projectionShell (V i) n)
    (hUinf : ∀ i, (⨅ n, U i n).starProjection =
      InnerProductSpace.rankOne ℂ (coordinateEmbedding i (xi i))
        (coordinateEmbedding i (xi i)))
    (hVinf : ∀ i, (⨅ n, V i n).starProjection =
      InnerProductSpace.rankOne ℂ (coordinateEmbedding o (xi o))
        (coordinateEmbedding o (xi o))) :
    ∃ L : I → HilbertSum H →L[ℂ] HilbertSum H,
      (∀ i, L i ∈ unitary (HilbertSum H →L[ℂ] HilbertSum H)) ∧
      (∀ i, L i (coordinateEmbedding i (xi i)) =
        coordinateEmbedding o (xi o)) ∧
      (∀ i n, (L i).comp (Submodule.projectionShell (U i) n) = W i n) ∧
      Representation.IsIrreducible
        (ambientInclusion (atomicRepresentation pi) L)
```

> Here `pi i` is $\pi_i$, `xi i` is $\xi_i$, and `coordinateEmbedding i (xi i)` is $b_i$. `U`,`V` are the two flags; `projectionShell (U i) n` is $P_{i,n} - P_{i,n+1}$. `hUinf`,`hVinf` supply the two rank-one defects. `W i n` is $W_{i,n}$ and `L i` is the completed unitary $L_i$. For these continuous operators, postfix `†` is the adjoint written as superscript $*$ in the prose. `ambientInclusion (atomicRepresentation pi) L` is the inclusion of this same generated $T$. The operators `S` and the commutant calculation occur in the linked proof, not as extra inputs in the signature.

## Lean realization notes

This theorem does not assert $L_o = 1$, simplicity of $T$, faithfulness of $\pi$, or capture of every irreducible representation of $T$. Its shell operators $W_{i,n}$ are not the cyclic-assembly map $W$ used in the later capture results. The star on operators is the Hilbert adjoint.

▼ **Exact Card identity**

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