Maximal faces of the C*-algebra unit ball and their unique support

`MathlibAnnex.Analysis.CStarAlgebra.isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall`

theorem

Characterizes maximal proper faces of a C*-algebra unit ball by extreme dual functionals.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity, and let $B_A = \{a \in A : \|a\| \leq 1\}$. Faces and convexity are taken over $\mathbb{R}$. We consider maximal proper faces of B_A.

Write A^* for the space of bounded complex-linear functionals on A, with the operator norm. For $\rho \in A^*$, write

$$F_\rho = \{a \in B_A : \rho(a) = 1\}.$$

For every subset $F \subseteq A$,

$$F \text{ is a maximal proper face of } B_A \iff F = F_\rho \text{ for some } \rho \in \operatorname{ext}_{\mathbb{R}}(B_{A^*}). \quad (1)$$

The supporting condition is the complex equality $\rho(a) = 1$. In the unit ball it is equivalent to the usual real supporting condition $\operatorname{Re}\rho(a) = 1$ when $\|\rho\| \leq 1$.

Assumptions

The source fixes the C*-order on A, which is used by the pure-state and interpolation inputs. Nontriviality is essential to the displayed equivalence: the zero algebra has no nonempty proper face, while its dual ball has the extreme point zero. No compactness of B_A, identity in A, or finite-dimensional assumption is imposed.

Conclusion

Equation (1) is the complete selected equivalence about subsets. The separately named theorem `supportFunctionals_unitBallSupportFace_eq_singleton` guarantees that the entire dual unit-ball support family of an extreme slice is the singleton containing that same extreme functional. `existsUnique_extreme_support_of_isMaximalFace` guarantees uniqueness. These are separate cited results, not extra conjuncts in the selected type.

Main citations

- Exact equivalence: `isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall`.

- Full support-family and unique-support results in the same module: `weakStarSupportFunctionals_unitBallSupportFace_eq_singleton`, `supportFunctionals_unitBallSupportFace_eq_singleton`, and `existsUnique_extreme_support_of_isMaximalFace` give the additional support and uniqueness guarantees just distinguished from (1).

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall (F : Set A) :
    NormedSpace.IsMaximalFace F ↔ ∃ rho : StrongDual ℂ A,
      rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ ∧
        F = NormedSpace.unitBallSupportFace ℂ rho
```

In the source	Mathematical meaning
Exact declaration `{A : Type u}`, the C*-algebra, order and nontriviality instances	The nonzero complex C*-algebra with its compatible order. The real convex structure is obtained by restricting scalars.
Exact declaration `(F : Set A)`	An arbitrary subset of the original algebra, not a set of states or a subset of the self-adjoint part.
Exact declaration `NormedSpace.IsMaximalFace F`	Nonemptiness, real convexity, the face property in the norm-closed unit ball, properness, and maximality among proper convex faces.
Exact declaration `∃ rho : StrongDual ℂ A`	One continuous complex-linear supporting functional is chosen for the face.
Exact declaration `rho ∈ (closedBall ... 1).extremePoints ℝ`	The same ρ is extreme for real convex combinations in the whole dual closed unit ball.
Exact declaration `F = NormedSpace.unitBallSupportFace ℂ rho`	Equality of the whole subset F with $\{a : \|a\| \le 1, \rho(a) = 1\}$. Together with extremality, this is the complete right side of the equivalence.
Exact declaration `↔` and the two conjuncts	Both directions hold, with one ρ satisfying both conjuncts. Support-family uniqueness is a separately named result, not a field hidden inside `IsMaximalFace`.

Exact surrounding source context (separate excerpts)

Source lines 34–34:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Source lines 35–35:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact declaration for the source guide

```
theorem isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall (F : Set A) :
    NormedSpace.IsMaximalFace F ↔ ∃ rho : StrongDual ℂ A,
      rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ ∧
        F = NormedSpace.unitBallSupportFace ℂ rho := by
```