The Hilbert space of a minimal-ideal compact model

`MathlibAnnex.Analysis.CStarAlgebra.MinimalClosedStarIdeal.CompactModelSpace`

abbrev

Names the cyclic Hilbert space chosen for a minimal closed ideal.

Statement

Let A be a complex C*-algebra with compatible order, and let $\pi : A \to \mathcal{B}(H)$ be a faithful representation with every value compact, on a complete complex Hilbert space. For a minimal nonzero closed $*$-ideal i of A, this abbreviation names the chosen cyclic Hilbert subspace $H_i \subseteq H$.

Definition

The minimal-ideal model construction chooses a pair $(p, x) \in A \times H$ with the following properties: p is a minimal projection, the closed ideal generated by p is the displayed ideal i, $x \neq 0$, $\pi(p)x = x$, and the same cyclic restriction $\sigma_{p,x} : I_p \to \mathcal{B}(H_i)$ is an exact compact-operator model: it is injective, every value is compact, and every compact operator on this same H_i is $\sigma_{p,x}(a)$ for some $a \in I_p$. Faithfulness and compact-valuedness alone do not include the last condition. These properties are guaranteed for the chosen pair; they are not additional assumptions.

Write I_p for the closed ideal generated by p. The chosen model subspace is

$$H_i = \overline{\{\pi(a)x : a \in I_p\}} \subseteq H$$

for that same pair (p, x), with its inherited norm and complex inner product. The closedness and Hilbert-space results ensure that H_i is complete and nonzero.

Assumptions

The algebra carries a partial order and compatible order. The input ideal is minimal among nonzero closed $*$-ideals. The representation π is injective and every value $\pi(a)$ is compact. There is no additional choice of Hilbert structure, no countability assumption and no global nonzero-algebra hypothesis.

Conclusion

The resulting space depends on the ideal and the chosen representation π. The compact-model isomorphism uses this same Hilbert space; different ideals need not give the same space.

Main citations

- Exact complete definition.
- MinimalClosedStarIdeal: the indexing type.

- compactModelWitness: one generator/vector pair and its full specification.
- IsCompactOperatorModel: the three conditions on one action.
- compactModelSubspace: the exact cyclic subspace.
- isClosed_compactModelSubspace and the completeness/nontriviality instances.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
abbrev CompactModelSpace (i : MinimalClosedStarIdeal A)
    (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsCompactlyRepresented) : Type v :=
  i.compactModelSubspace pi hpi
```

In the source	Mathematical meaning
Exact declaration `i : MinimalClosedStarIdeal A`	A minimal nonzero closed $*$-ideal.
Exact declaration `pi` and `hpi`	The fixed representation and its faithful compact-valued condition, used to choose the projection and cyclic vector.
Exact declaration `: Type v`	The resulting space H_i is a subspace of the given Hilbert space H.
Exact declaration `i.compactModelSubspace pi hpi`	The closed cyclic subspace associated with the chosen pair (p, x), with norm and inner product inherited from H.

Exact surrounding source context (separate excerpts)

Source lines 128–129:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Source lines 140–140:

```
namespace MinimalClosedStarIdeal
```