MATHLIBANNEX / CANONICAL DECLARATION CARD

Closed-ideal simplicity of the shell-generated algebra

`MathlibAnnex.CStarAlgebra.CAR.isSimpleCStarAlgebra_shellFamilyTarget`

theorem

Uses a faithful irreducible representation in the unique unitary-equivalence class to rule out proper nonzero closed two-sided ideals.

Statement

For the fixed algebra T described below, every norm-closed two-sided ideal $\mathfrak{a}$ satisfies $\mathfrak{a} = \{0\}$ or $\mathfrak{a} = T$. The algebra is nonzero, so it is simple as a C*-algebra.

Assumptions

Let A be the completed CAR algebra with matrix stages $M_{2^n}(\mathbb{C})$ embedded by $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit, $p_n = f_n - f_{n+1}$, and ω for the root state characterized by $\omega(\iota_n(a)) = a_{00}$ on each stage, where ι_n is its canonical embedding.

Index chosen pure-state Gelfand-Naimark-Segal (GNS) representations by their unitary-equivalence classes $j \in I$, with root index o and representative states φ_j.

A fixed shell family consists of complex-linear unital star automorphisms α_j and elements $w_{j,n} \in A$ satisfying $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$.

At the root, α_o is the identity and $w_{o,n} = p_n$.

Let $\mathcal{H} = \bigoplus_{j \in I} H_j$ be the Hilbert direct sum of these GNS spaces, $\pi = \bigoplus_j \pi_j$ their representation of A, and $b_j = e_j \xi_j \in \mathcal{H}$ its embedded unit cyclic vector, where $\xi_j \in H_j$ is the selected unit cyclic vector and $e_j : H_j \to \mathcal{H}$ is coordinate inclusion.

Choose once a family of unitary operators $L_j \in B(\mathcal{H})$ from the construction of the shell unitaries: $L_j b_j = b_o$, $L_j \pi(\alpha_j(p_n)) = \pi(w_{j,n})$, and $L_o = I_{\mathcal{H}}$. Here $B(\mathcal{H})$ denotes the bounded complex-linear operators on $\mathcal{H}$.

Put $T = C^*(\pi(A), \{L_j : j \in I\}) \subseteq B(\mathcal{H})$, the norm-closed unital star algebra generated by these operators. Let $\iota : A \to T$ be $\iota(a) = \pi(a)$ with its codomain restricted to T, and let $\rho_0 : T \to B(\mathcal{H})$ be inclusion.

The ideals in this assertion are two-sided algebraic ideals whose underlying sets are norm closed. The family is supplied as input. The shell-model realization theorem provides the faithful source representation and irreducible inclusion; the universal capture theorem identifies every irreducible class with that inclusion.

Conclusion

The conclusion is simplicity with respect to closed two-sided ideals. It is not a claim that every algebraic ideal, with no closure hypothesis, is trivial.

Proof route

Let $\mathfrak{a}$ be proper and closed. The ideal-separating pure-state GNS argument in the closed-ideal simplicity criterion gives an irreducible representation that annihilates $\mathfrak{a}$. It is unitarily equivalent to the faithful inclusion ρ_0. For $t \in \mathfrak{a}$, the intertwining equation therefore makes $\rho_0(t)$ zero. Since ρ_0 is injective, $t = 0$. Thus every proper closed ideal is zero.

Proof steps

1. If $\mathfrak{a} = T$ there is nothing to prove. Otherwise use a pure GNS representation σ with $\sigma(t) = 0$ for every $t \in \mathfrak{a}$.
2. Choose the unitary U from the faithful inclusion to σ. For $t \in \mathfrak{a}$, $U\rho_0(t)x = \sigma(t)Ux = 0$ for all x.
3. Injectivity of U gives $\rho_0(t) = 0$; injectivity of ρ_0 gives $t = 0$. Nonzeroness of T follows from its embedded copy of A.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicShellModel · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isUniqueIrreducibleModel_shellFamilyInclusion · Exact source
- MathlibAnnex.CStarAlgebra.CAR.shellFamilySourceHom_injective · Exact source
- MathlibAnnex.CStarAlgebra.isSimpleCStarAlgebra_of_uniqueIrreducibleModel · Exact source

Lean source signature (exact)

```
theorem isSimpleCStarAlgebra_shellFamilyTarget (family : RepresentativeShellFamily) :
    MathlibAnnex.CStarAlgebra.IsSimpleCStarAlgebra (ShellFamilyTarget family)
```

> Here `ShellFamilyTarget family` is T. `IsSimpleCStarAlgebra` means that T is nonzero and every norm-closed two-sided ideal is either zero or the whole algebra. This is exactly the closed-ideal assertion above; no condition on arbitrary nonclosed ideals is added.

Lean realization notes

The proof uses capture for the Hilbert spaces of canonical pure GNS representations, which lie in the source-algebra universe. The stronger universe-polymorphic capture result supplies this case. Faithfulness is essential: uniqueness of an irreducible class without a faithful representation would not justify this argument.

▼ **Exact Card identity**

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