The primitive spectrum and the primitive geometric spectrum

`MathlibAnnex.Analysis.CStarAlgebra.primitiveSpectrumHomeomorph`

def

Identifies the primitive ideal spectrum with the primitive geometric spectrum.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. There is a specified ordinary homeomorphism

$$\Psi_A : \mathrm{Prim}(A) \longrightarrow \mathrm{PS}(A),$$

from the independently defined primitive ideal spectrum to the primitive geometric spectrum. Both spaces have points that are subsets of A, and this homeomorphism retains each such subset. The two spectra have the hull-kernel topologies described below.

Definition

Write $\mathrm{Prim}(A)$ for the primitive spectrum: its points are subsets $P = \ker \pi_\varphi \subseteq A$, where φ is a pure norm-one positive bounded complex-linear functional and π_φ is its GNS representation of the original algebra. This is a representation kernel, not the zero set of φ.

For the geometric side, let $S(A)$ be the family of kernels of extreme points of the closed unit ball of the Banach dual of A, with convexity over $\mathbb{R}$. The identification of this family with the face-and-support geometric structure is the conclusion of the extreme-functional kernel characterization. Its closure is $c_A(\mathcal{U}) = \{I \in S(A) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$. Components use the equivalence relation generated by equality and nonclosed two-point sets. For a component C, put $J_C = \bigcap_{I \in C} I$ and $\mathrm{PS}(A) = \{J_C : C \text{ a component}\}$, with repeated subsets identified.

The source topology on $\mathrm{Prim}(A)$ and target topology on $\mathrm{PS}(A)$ have the hull-kernel closures

$$\overline{\mathcal{U}} = \left\{ P : \bigcap_{Q \in \mathcal{U}} Q \subseteq P \right\}, \qquad \overline{\mathcal{V}} = \left\{ J : \bigcap_{K \in \mathcal{V}} K \subseteq J \right\}.$$

The ambient space in each formula is the indicated spectrum; an empty intersection is A.

The map Ψ_A sends a primitive kernel to the identical subset regarded as a primitive geometric point; its inverse also retains the subset. The subset-preserving closure homeomorphism supplies this bijection with closure continuity in both directions. The primitive-spectrum topologizability

theorem and geometric-spectrum topologizability theorem supply ordinary topologies with exactly these closures, making the same bijection an ordinary homeomorphism.

Assumptions

The input is a nonzero complex C*-algebra, not necessarily unital, with the compatible partial order and star order in the displayed scope. Both spectra carry the hull-kernel topologies stated above. The correspondence of the two families and the topologizability results are provided by the cited source declarations, not additional assumptions on A.

Conclusion

The output is the ordinary homeomorphism Ψ_A with the same underlying subset in both directions. It includes continuity of that map and its inverse. It does not identify a functional kernel with a representation kernel.

Main citations

- Exact full definition: `primitiveSpectrumHomeomorph`.
- Primitive family equality and SpectralComparison.lean, `primitiveSpectrumEquivGeometric`: the forward and inverse carrier maps.
- Spectrum/Topology.lean: the independent hull–kernel topology and its finite-union proof.
- SpectralHomeomorph.lean, `primitiveSpectrumClosureHomeomorph`, `image_primitiveSpectrumClosure`, `image_symm_primitiveGeometricSpectrumClosure`: the two closure directions and their ordinary topological conversion.
- Ryotaro Tanaka, arXiv:2504.05551v1, Theorem 5.11 and Corollary 5.12: the primitive-spectrum comparison and topologizability.
- primitiveSpectrumClosureHomeomorph: the subset-preserving closure homeomorphism.
- isTopologizable_primitiveGeometricSpectrumClosure: the target topology guarantee.

Lean source signature (exact)

The full definition below is copied from the pinned source.

```
def primitiveSpectrumHomeomorph :
    @_root_.Homeomorph (PrimitiveSpectrum A) (NormedSpace.PrimitiveGeometricSpectrum ℂ A)
      inferInstance (primitiveGeometricSpectrumTopology A) :=
  (primitiveSpectrumClosureHomeomorph A).toTopologicalHomeomorph
    (isTopologizable_primitiveSpectrumClosure A)
    (isTopologizable_primitiveGeometricSpectrumClosure A)
```

In the source	Mathematical meaning
Exact declaration `A` and the surrounding instances	The same nonzero complex C*-algebra with compatible order; a unit is not required.
Exact declaration `PrimitiveSpectrum A` and `primitiveGeometricSpectrumTopology A`	The representation-kernel spectrum and the stated hull-kernel topology on the component-intersection family.
Exact declaration `primitiveSpectrumClosureHomeomorph A`	The same subset-preserving bijection with full closure continuity in both directions.
Exact declaration `.toTopologicalHomeomorph` and the two `isTopologizable` inputs	The same bijection is a homeomorphism for the two ordinary topologies with the specified closures.

Exact surrounding source context (separate excerpts)

Source lines 28–28:

```
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```

Source lines 29–29:

```
variable [Nontrivial A]
```