The atomic common range is one embedded GNS line

`MathlibAnnex.CStarAlgebra.CAR.iInf_range_atomic_transportedFlag_eq_span`

theorem

Assembles the matching and inequivalent fiber calculations in an arbitrary Hilbert direct sum.

Statement

For a fixed shell family and $i \in I$ as below, let $\mathcal{H} = \bigoplus_{j \in I} H_j$ and $\pi = \bigoplus_{j \in I} \pi_j$. Denote the coordinate embedding by $e_i : H_i \to \mathcal{H}$. Then $\bigcap_{n \geq 0} \operatorname{ran} \pi(f_{i,n}) = \mathbb{C} e_i(\xi_i)$, where $f_{i,n} = \alpha_i(f_n)$.

Assumptions

Let A be the completed CAR algebra, the completion of the matrix stages $M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit. These are decreasing projections with $f_0 = 1$. Write ι_n for the canonical embedding of stage n. The root state ω is determined by $\omega(\iota_n(a)) = a_{00}$ on every stage.

Choose one pure state φ_j from each unitary-equivalence class $j \in I$ of pure-state Gelfand-Naimark-Segal (GNS) representations, retaining ω at the root index o. Write (H_j, π_j, ξ_j) for its complete complex GNS space, unital star representation and unit cyclic vector. Thus $\varphi_j(a) = \langle \xi_j, \pi_j(a)\xi_j \rangle$, and the vectors $\pi_j(a)\xi_j$ are dense in H_j. Inner products are linear in the second argument.

A fixed shell family supplies unital complex star automorphisms α_j and elements $w_{j,n} \in A$ with $\varphi_j \circ \alpha_j = \omega$, $w_{j,n}^* w_{j,n} = \alpha_j(f_n - f_{n+1})$, and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$. Fix $i \in I$ and put $f_{i,n} = \alpha_i(f_n)$. The family is supplied; its existence is not an extra conclusion here. The Hilbert direct sum consists of square-summable families $x = (x_j)_{j \in I}$, and $\pi(a)$ acts coordinatewise. No countability of I is assumed. The coordinate embedding is isometric, so $e_i(\xi_i)$ is a unit vector.

Conclusion

The orthogonal projection onto this common range is the rank-one map $x \mapsto \langle e_i(\xi_i), x \rangle e_i(\xi_i)$. The decreasing-projection theorem therefore gives strong convergence of $\pi(f_{i,n})$ to this map. In the construction of the shell unitaries, this is the one-dimensional residual space joined to the root residual line when the difference-shell maps are completed to a unitary.

Proof route

A vector is in the range of an orthogonal projection exactly when that projection fixes it. Thus a vector $x \in \mathcal{H}$ belongs to the common range exactly when every coordinate x_j is fixed by all $\pi_j(f_{i,n})$. The matching-fiber common-range theorem forces x_i to be a scalar multiple of ξ_i, while the inequivalent-fiber common-range theorem forces $x_j = 0$ for $j \neq i$. Conversely, the embedded vector $e_i(\xi_i)$ and all its scalar multiples are fixed by every $\pi(f_{i,n})$.

Proof steps

1. For a vector in the common range, apply each coordinate evaluation to $\pi(f_{i,n})x = x$. This gives fixedness in every fiber without a sum-limit argument.

2. The common projection in fiber i sends x_i to $\langle \xi_i, x_i \rangle \xi_i$; since x_i is fixed it equals that value. For $j \neq i$, the common projection is zero, so fixedness gives $x_j = 0$. Therefore $x = \langle \xi_i, x_i \rangle e_i(\xi_i)$.

3. Every flag projection fixes ξ_i in the matching fiber. Acting coordinatewise therefore fixes every scalar multiple of $e_i(\xi_i)$, proving the reverse inclusion. The unit norm then identifies the projection onto the resulting line.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selected_commonFixedProjection_eq_rankOne · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selected_commonFixedProjection_eq_zero · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.iInf_range_atomicRepresentation_eq_span · Exact source
- Submodule.tendsto_starProjection_iInf · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicShellModel · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem iInf_range_atomic_transportedFlag_eq_span
    (family : RepresentativeShellFamily) (i : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit) :
    (⨅ n, (atomicRepresentation
      (MathlibAnnex.CStarAlgebra.PureState.selectedRepresentation completedRootPureState)
      (transportedFlag family i n)).range) =
      ℂ ∙ MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState i
        (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState i)
```

> Here `Limit` is A and `completedRootPureState` is the root state ω, bundled with its purity proof. `atomicRepresentation` is the coordinatewise representation π on $\mathcal{H}$. The infimum `⨅ n` is the intersection of the represented ranges. `selectedEmbedding completedRootPureState i` is e_i, and `selectedVector completedRootPureState i` is ξ_i. The right side `ℂ ∙ ...` is its complex linear span, not the entire i-th GNS fiber.

Lean realization notes

The exact declaration is the equality of subspaces, and its identity remains separate from the two fiber declarations. The rank-one and strong-limit assertions are explained consequences. The coordinate proof works for an arbitrary index set; it does not exchange an uncountable sum with a limit or assert operator-norm convergence.

▼ **Exact Card identity**

Language: en
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