Extending the CAR trace to the fixed shell target

`MathlibAnnex.CStarAlgebra.CAR.exists_state_extension_trace`

theorem

The source trace first becomes a state on the existing target algebra.

Statement

Let A be the completed CAR algebra, the completion of the matrix system $M_{2^n}(\mathbb{C})$, and let τ be its normalized trace. Fix a representative shell family: for each chosen pure-GNS class j it specifies an automorphism α_j and partial isometries $w_{j,n}$ between the transported and root shells. Fix the links L_j chosen by this family in the selected atomic representation π. Throughout this Card,

$$T = C^*(\pi(A), \{L_j\}), \qquad \iota : A \longrightarrow T, \quad \iota(a) = \pi(a).$$

Thus T is the already constructed concrete unital C*-algebra, and ι is its injective source map. The links and target are kept fixed. A state means a positive continuous complex linear functional taking 1 to 1. There exists a state ϕ on T satisfying $\phi(\iota(a)) = \tau(a)$ for every $a \in A$.

Assumptions

The representative shell family is fixed. No representation of the target is an additional input to this extension theorem.

Conclusion

The set of state extensions of τ to this same T is nonempty. Traciality on all of T and uniqueness of the extension are separate results.

Proof route

Transfer τ to the isometric copy $\iota(A)$, extend the bounded functional, and use normalization to obtain positivity.

Proof steps

1. The injective unital C*-homomorphism ι is isometric. Consequently $\lambda(\iota(a)) = \tau(a)$ is well-defined and satisfies

   $$|\lambda(\iota(a))| \leq \|a\| = \|\iota(a)\|, \qquad \lambda(1) = 1.$$

   The input bounds are `norm_trace_le` and `trace_one`, and injectivity is `shellFamilySourceHom_injective`.

2. Apply the state-extension theorem for an injective unital star homomorphism, `exists_state_extension_of_injective`, with source functional τ. Its Hahn–Banach step extends λ with norm at most 1; its normalized-contraction positivity argument makes the extension a state. Its restriction equality gives exactly $\phi \circ \iota = \tau$.

Main citations

- The stated existence or structural result · Exact source
- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- State extension along an injective unital embedding · Exact source
- Normalization of the CAR trace · Exact source
- Contractive bound on the CAR trace · Exact source

Lean source signature (exact)

```
theorem exists_state_extension_trace (family : RepresentativeShellFamily) :
    ∃ ϕ : ShellFamilyTarget family →L[ℂ] ℂ,
      ϕ ∈ MathlibAnnex.Analysis.CStarAlgebra.stateSpace (ShellFamilyTarget family) ∧
      ∀ b, ϕ (shellFamilySourceHom family b) = trace b
```

> Here `Limit`, `trace`, `ShellFamilyTarget family`, and `shellFamilySourceHom family` are A, τ, T, ι. The quantified ϕ is a state on T, and the final `∀ b` is equality on every source element. The linked proof invokes `exists_state_extension_of_injective`; that supplier is named in the proof, not in the displayed theorem signature.

Lean realization notes

The extension theorem supplies an existence witness. The later definition `traceExtension` chooses one such witness; it does not change T or create a universal completion.

▼ **Exact Card identity**

Language: en

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