The atomic-shell construction on selected pure-GNS fibers

`MathlibAnnex.CStarAlgebra.AtomicConstruction.exists_irreducible_pureAtomicShellModel`

theorem

Pure-state GNS data supplies the irreducible and inequivalent fibers required by the generic shell construction.

Statement

Let A be a unital complex C*-algebra, with the positive order used for its states, and fix a pure state ω. A pure state is an extreme point of the normalized positive continuous linear functionals. Let I be the set of pure-state GNS representations modulo unitary equivalence. For each $j \in I$, choose a pure state φ_j in that class, retaining $\varphi_o = \omega$ literally at $o = [\omega]$. Write (H_j, π_j, ξ_j) for its GNS space, representation and canonical cyclic vector. Write $\mathcal{H} = \bigoplus_{j \in I} H_j$ for the Hilbert direct sum, $e_j : H_j \to \mathcal{H}$ for coordinate inclusion, and $\pi(a)(x_j)_j = (\pi_j(a)x_j)_j$. Here the sum consists of square-summable families and has no countability restriction on I. Put $b_j = e_j\xi_j$. Given the shell data specified below, there exist unitaries $L_j \in B(\mathcal{H})$ with $L_j b_j = b_o$ and $L_j(P_{j,n} - P_{j,n+1}) = W_{j,n}$. The inclusion of $T = C^*(\pi(A), \{L_j : j \in I\})$ into $B(\mathcal{H})$ is irreducible.

Assumptions

For each $i \in I$, let $(U_{i,n})_{n \geq 0}$ and $(V_{i,n})_{n \geq 0}$ be decreasing subspaces of $\mathcal{H}$, with $U_{i,0} = V_{i,0} = \mathcal{H}$. All these subspaces and both intersections admit orthogonal projections. Write $P_{i,n} = P_{U_{i,n}}$, $Q_{i,n} = P_{V_{i,n}}$, and $P_{i,\infty} = P_{\bigcap_n U_{i,n}}$, $Q_{i,\infty} = P_{\bigcap_n V_{i,n}}$. The given bounded shell maps $W_{i,n}$ satisfy $W_{i,n}^* W_{i,n} = P_{i,n} - P_{i,n+1}$ and $W_{i,n} W_{i,n}^* = Q_{i,n} - Q_{i,n+1}$. Their limiting defects are prescribed: $P_{i,\infty} = \theta_{b_i,b_i}$ and $Q_{i,\infty} = \theta_{b_o,b_o}$. Here $\theta_{a,b}x = \langle b, x\rangle a$, with the inner product linear in its second variable. In these formulas the generic index i ranges over the GNS classes I and the vectors are the selected GNS vectors just defined. The two flags, shell maps, support equations and limiting rank-one equations remain hypotheses; purity alone does not supply them.

Conclusion

One family of unitary links realizes all prescribed shell actions and joins every selected cyclic vector to the root vector, with irreducible concrete target action. This is the specialization of the generic atomic-shell result to the chosen pure-GNS family.

Proof route

Use the selected GNS representations as the fibers of the generic atomic-shell theorem, with root class $o = [\omega]$ and the same selected vectors ξ_j. Three proved pure-GNS facts discharge its representation hypotheses; all shell hypotheses pass through unchanged.

Proof steps

1. The selected-vector normalization theorem gives $\|\xi_j\| = 1$, so each H_j is nonzero. The selected-vector functional equals the pure state φ_j, and the selected GNS orbit is dense. The separate pure-GNS irreducibility theorem uses these facts to show that each π_j is irreducible.

2. For $i \neq j$, the selected-class non-intertwining theorem rules out a unitary intertwiner between π_i and π_j. The GNS classes, not merely the state functionals, are required to be distinct. Thus the pairwise-unitary-inequivalence hypothesis of the generic theorem is satisfied.

3. Apply the generic atomic-shell theorem with these fibers, the distinguished index o, the already fixed vectors ξ_j, and the given W, U, V. The displayed support and rank-one intersection equations are the remaining inputs. Coordinate inclusion here is exactly e_j, so its output vector-link and shell equations are the required ones, with no new choice of GNS representatives.

Main citations

- The stated existence or structural result · Exact source
- Generic atomic-shell construction with its full input hypotheses · Exact source
- Irreducibility of each selected GNS fiber · Exact source
- Inequivalence of distinct selected classes · Exact source
- Norm one of the selected cyclic vectors · Exact source
- Selected coordinate inclusions · Exact source

Lean source signature (exact)

```
theorem exists_irreducible_pureAtomicShellModel
    (root : PureState A)
    (W : PureState.GNSClass A → ℕ →
      PureState.SelectedAtomicHilbert root →L[ℂ]
        PureState.SelectedAtomicHilbert root)
    (U V : PureState.GNSClass A → ℕ →
      Submodule ℂ (PureState.SelectedAtomicHilbert root))
    [∀ i n, (U i n).HasOrthogonalProjection]
    [∀ i n, (V i n).HasOrthogonalProjection]
    [∀ i, (⨅ n, U i n).HasOrthogonalProjection]
    [∀ i, (⨅ n, V i n).HasOrthogonalProjection]
    (hU : ∀ i, Antitone (U i)) (hV : ∀ i, Antitone (V i))
    (hU0 : ∀ i, U i 0 = ⊤) (hV0 : ∀ i, V i 0 = ⊤)
    (hInitial : ∀ i n, ((W i n)†).comp (W i n) =
      Submodule.projectionShell (U i) n)
    (hFinal : ∀ i n, (W i n).comp ((W i n)†) =
      Submodule.projectionShell (V i) n)
    (hUinf : ∀ i, (⨅ n, U i n).starProjection =
      InnerProductSpace.rankOne ℂ
        (PureState.selectedEmbedding root i (PureState.selectedVector root i))
        (PureState.selectedEmbedding root i (PureState.selectedVector root i)))
    (hVinf : ∀ i, (⨅ n, V i n).starProjection =
      InnerProductSpace.rankOne ℂ
        (PureState.selectedEmbedding root root.classOf
          (PureState.selectedVector root root.classOf))
        (PureState.selectedEmbedding root root.classOf
          (PureState.selectedVector root root.classOf))) :
    ∃ L : PureState.GNSClass A →
        PureState.SelectedAtomicHilbert root →L[ℂ]
          PureState.SelectedAtomicHilbert root,
      (∀ i, L i ∈ unitary
        (PureState.SelectedAtomicHilbert root →L[ℂ]
          PureState.SelectedAtomicHilbert root)) ∧
      (∀ i, L i (PureState.selectedEmbedding root i
          (PureState.selectedVector root i)) =
        PureState.selectedEmbedding root root.classOf
          (PureState.selectedVector root root.classOf)) ∧
      (∀ i n, (L i).comp (Submodule.projectionShell (U i) n) = W i n) ∧
      Representation.IsIrreducible
        (ambientInclusion
          (atomicRepresentation (PureState.selectedRepresentation root)) L)
```

> Here `root` is ω, `PureState.GNSClass A` is I, and `SelectedAtomicHilbert root` is $\mathcal{H}$. `selectedEmbedding root i (selectedVector root i)` is b_i; the root expression is b_o. The signature retains `hInitial`,`hFinal`,`hUinf`,`hVinf` as shell hypotheses. The linked proof supplies `isIrreducible_selectedRepresentation`, `no_unitaryIntertwiner_selectedRepresentation` and `norm_selectedVector` to the generic theorem; those are separately cited, rather than assumed to follow from the names of the types.

Lean realization notes

The selected vectors are fixed before applying the construction and are not replaced by arbitrary unit vectors. No target capture or simplicity conclusion is part of this result, and the selected representations are not assumed faithful. No countability or separability of I or $\mathcal{H}$ is required.

▼ **Exact Card identity**

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Card revision: 1 · SHA-256: e244b62ef37cd89bb88e21906ffcf366d76ae4d38f5c2da2ce82cef0f7bb177b
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