The primitive geometric spectrum

`NormedSpace.PrimitiveGeometricSpectrum`

def

The primitive geometric spectrum

Statement

The primitive geometric spectrum records the common intersections of the points belonging to each component of the geometric structure space. A primitive geometric point is an actual subset of the original normed space. Repeated intersections give one point, even if they arise from different components.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, with closed unit ball $B_E = \{x \in E : \|x\| \le 1\}$ and continuous $\mathbb{K}$-linear dual E^*. For each nonempty maximal proper real convex face M of B_E, define

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) &= \{f \in E^* : \|f\| \le 1,\ f|_M = 1\}, \\ I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f. \end{aligned}$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

A face is a convex subset containing both endpoints of every segment in the ball whose interior meets it; maximality is among proper convex faces. Let $S_{\mathbb{K}}(E)$ be the family of the resulting subsets $I_{\mathbb{K}}(M)$, equipped with the following operation on $\mathscr{A} \subseteq S_{\mathbb{K}}(E)$:

$$c_E(\mathscr{A}) = \left\{ I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I \right\}.$$

The empty intersection is E. Two geometric points I, J are linked when $I = J$ or $c_E(\{I, J\}) \neq \{I, J\}$. Their finite-chain equivalence relation is denoted by $\sim_{c_E}$, and

$$C_E(I) = \{J \in S_{\mathbb{K}}(E) : I \sim_{c_E} J\}$$

is the component of I. For a component set C, put

$$J_C = \bigcap_{I \in C} I.$$

Define the **primitive geometric spectrum** by

$$\mathrm{PS}_{\mathbb{K}}(E) = \{J_{C_E(I)} : I \in S_{\mathbb{K}}(E)\}.$$

This is a family of subsets of E. A vector x belongs to J_C precisely when it belongs to every geometric point in C. All intersections are set intersections; no subspace or ideal structure is assumed.

Assumptions

The space E is normed over $\mathbb{K}$, without a completeness or nontriviality hypothesis. The component construction uses the stated hull–kernel closure. Neither closedness of the component sets nor injectivity of the assignment $C \mapsto J_C$ is assumed.

Conclusion

The resulting carrier contains exactly the component intersections. The cited properness result follows from

$$I \in C_E(I) \quad \Longrightarrow \quad J_{C_E(I)} \subseteq I \neq E.$$

Thus no primitive geometric point is all of E. This permits the separately defined hull–kernel closure: for $\mathscr{T} \subseteq \mathrm{PS}_{\mathbb{K}}(E)$,

$$d_E(\mathscr{T}) = \left\{ J \in \mathrm{PS}_{\mathbb{K}}(E) : \bigcap_{J' \in \mathscr{T}} J' \subseteq J \right\}.$$

The present declaration supplies the carrier; the cited closure declaration supplies d_E. For the zero space the carrier is empty. The empty-intersection convention therefore does not introduce an extra primitive point in that case.

Main citations

- Geometric points and their closure: `NormedSpace.geometricStructureFamily` and `NormedSpace.geometricStructureClosure`, lines 37–65; maximal faces, `NormedSpace.IsMaximalFace`, lines 32–39.
- Finite-chain components: `ClosureSpace.component` and its generating relation, lines 30–71.
- Intersections and properness: `ClosureSpace.HullKernel.componentKernel_ne_univ`, lines 35–50; the intersection family is defined at lines 72–75.
- Exact carrier and range description: `NormedSpace.PrimitiveGeometricSpectrum`, lines 48–54; the range characterization, lines 93–101.

- Closure and emptiness: `NormedSpace.primitiveGeometricSpectrumClosure`, lines 109–121; the zero-space case, lines 179–199.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 4.6 and Theorem 4.7: component intersections and their closure for nontrivial Banach spaces.

Lean source signature (exact)

The scalar and normed-space declarations shared by this definition are reproduced separately below.

```
def PrimitiveGeometricSpectrum : Type u :=
  {J : Set E // J ∈ primitiveGeometricSpectrumFamily 𝕜 E}
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$, possibly zero and not necessarily complete.
Exact declaration `J : Set E`	A subset $J \subseteq E$, rather than a subset of the geometric point set.
Exact declaration `J ∈ primitiveGeometricSpectrumFamily 𝕜 E`	There exists a geometric point $I \in S_{\mathbb{K}}(E)$ with $J = J_{C_E(I)} = \bigcap_{K \in C_E(I)} K$. The component $C_E(I)$ is formed using c_E.
Exact declaration `{J : Set E // J ∈ primitiveGeometricSpectrumFamily 𝕜 E}`	The full RHS defines $\mathrm{PS}_{\mathbb{K}}(E)$ as exactly these intersection subsets with their membership condition. The underlying subset is `J.val`; neither a representative point nor a component witness is stored.
Exact declaration `: Type u`	The output is the primitive geometric carrier. Equality is equality of subsets of E, so equal intersections give one point; no ideal structure or injectivity of the component-intersection assignment is asserted.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def PrimitiveGeometricSpectrum : Type u :=
  {J : Set E // J ∈ primitiveGeometricSpectrumFamily 𝕜 E}

/-- Send a geometric-structure point to its component. -/
```