Pure CAR states admit two-sided inner intertwining sequences

`MathlibAnnex.CStarAlgebra.CAR.hasInnerIntertwiningSequence_of_pure`

theorem

Finite-set vector-state transport supplies explicit corrections; their summable conjugation bounds and state estimates yield the required two-sided sequence.

Statement

Let A be the completed CAR algebra, the norm completion of the matrix stages $A_n = M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$, with the fixed coordinate reindexing. A state means a positive continuous complex-linear functional taking value 1 at the identity; a pure state is an extreme point of this convex state space. For a unitary $u \in A$, write $\mathrm{Ad}(u)(a) = uau^*$. For every pair of pure states φ, ψ on A, there is a sequence $\alpha_n = \mathrm{Ad}(u_n)$ such that, for every $a \in A$, both $(\alpha_n(a))$ and $(\alpha_n^{-1}(a))$ are norm-Cauchy, and $\varphi(\alpha_n(a)) \to \psi(a)$.

Assumptions

The input functionals are pure states on the fixed completed CAR algebra. Their GNS representations and unit cyclic vectors are constructed from these states; no identification of their Hilbert spaces, equivalence of their representations, or initial closeness of the two states is assumed.

Conclusion

The required inner intertwining sequence exists for every pure-state pair. Its inverse condition concerns the actual inverses of the produced automorphisms. This is the supplier used by the separate limit theorem.

Proof route

Use the finite-set vector-state transport theorem to construct unitaries l_n, r_n. Put $L_n(a) = l_n^* a l_n$, $R_n(a) = r_n^* a r_n$, and choose a dense sequence (d_j). The corrections are chosen to give, for $j \le n$,

$$\|Q_{n+1}(d_j) - Q_n(d_j)\| < 2^{-n} \quad (Q = L, L^{-1}, R, R^{-1}),$$
$$|\varphi(L_{n+1}(R_n^{-1}(d_j))) - \psi(d_j)| < 2^{-n}.$$

The first estimates make the four sequences pointwise Cauchy. The second gives the state convergence for $\alpha_n = L_{n+1} \circ R_n^{-1} = \mathrm{Ad}(l_{n+1}^* r_n)$. The construction below records the local theorem's hypothesis, the correction it supplies, and the inequality that enables the next application.

Proof steps

1. **The local estimate used in every correction.** Put $\theta_v(a) = v^* a v$. Fix an irreducible representation π, a finite set $P \subset A$, and $\varepsilon > 0$. The finite-set transport theorem gives a finite matrix-unit test set T and $\delta > 0$. If μ is a unit vector state of π and ν is a unit vector state of another representation, its hypothesis is

$$|\mu(t) - \nu(t)| < \delta \qquad (t \in T).$$

Under this hypothesis, **for every** finite $F \subset A$ and $\gamma > 0$, it gives a unitary v such that

$$\begin{aligned} |\mu(\theta_v(b)) - \nu(b)| &< \gamma \qquad (b \in F), \\ \|\theta_v(a) - a\| &< \varepsilon \qquad (a \in P), \\ \|\theta_v^{-1}(a) - a\| &< \varepsilon \qquad (a \in P). \end{aligned}$$

Here $\mu \circ \theta_v$ is the state of $\pi(v)\zeta$ when μ is the state of ζ. The tests T, δ are chosen before F, γ. The path supplied by the theorem is not needed for this sequence conclusion.

2. **The quantities to be constructed and the induction hypothesis.** Take the irreducible GNS representations π_φ, π_ψ with unit cyclic vectors ξ, η. Choose a dense sequence (d_j) in A, and put $D_n = \{d_0, \ldots, d_n\}$ and $\varepsilon_n = 2^{-n}$. For unitaries l_n, r_n, write

$$\begin{aligned} L_n(a) &= l_n^* a l_n, \qquad & R_n(a) &= r_n^* a r_n, \\ \varphi_n &= \varphi \circ L_n, \qquad & \psi_n &= \psi \circ R_n. \end{aligned}$$

These are the vector states of $\pi_\varphi(l_n)\xi$ and $\pi_\psi(r_n)\eta$. To control both an automorphism and its inverse, use

$$P_n^\varphi = D_n \cup L_n^{-1}(D_n), \qquad P_n^\psi = D_n \cup R_n^{-1}(D_n).$$

Apply the local theorem to $(\pi_\varphi, P_n^\varphi, \varepsilon_n)$, obtaining $T_n^\varphi, \delta_n^\varphi$. The induction hypothesis is the explicit inequality

$$\boxed{|\varphi_n(t) - \psi_n(t)| < \delta_n^\varphi \quad (t \in T_n^\varphi).}$$

It enables the local theorem to change φ_n toward ψ_n with any later finite request. Purity is used to make **both** GNS representations irreducible; they will be used in separate applications of the theorem.

3. **Initialize the induction without assuming that φ and ψ are close.** First obtain $\widehat{T}, \hat{\delta}$ from the local theorem for $(\pi_\psi, D_0, \varepsilon_0)$. The unrestricted approximation theorem supplies l_0 with

$$|\varphi(\theta_{l_0}(t)) - \psi(t)| < \hat{\delta} \qquad (t \in \widehat{T}).$$

Now $L_0 = \theta_{l_0}$ is fixed. Obtain $T_0^\varphi, \delta_0^\varphi$ for $(\pi_\varphi, P_0^\varphi, \varepsilon_0)$. The displayed inequality enables the π_ψ theorem with $\mu = \psi$ and $\nu = \varphi_0$. Request $F = T_0^\varphi$ and $\gamma = \delta_0^\varphi$. Its output r_0 satisfies

$$|\psi(\theta_{r_0}(t)) - \varphi_0(t)| < \delta_0^\varphi \qquad (t \in T_0^\varphi).$$

With $R_0 = \theta_{r_0}$, this is exactly the induction hypothesis at $n = 0$.

4. **Construct l_{n+1} and obtain the estimate that will imply state convergence.** Assume the induction hypothesis at n. First obtain T_n^ψ, δ_n^ψ from the local theorem for $(\pi_\psi, P_n^\psi, \varepsilon_n)$. Apply the already enabled π_φ theorem to $\mu = \varphi_n, \nu = \psi_n$, with

$$F = T_n^\psi \cup R_n^{-1}(D_n), \qquad \gamma = \min\{\delta_n^\psi, \varepsilon_n\}.$$

Let v_n be its output, and define

$$l_{n+1} = v_n l_n, \qquad L_{n+1} = L_n \circ \theta_{v_n}, \qquad \varphi_{n+1} = \varphi_n \circ \theta_{v_n}.$$

The theorem gives the two estimates

$$\begin{aligned} \|\theta_{v_n}^{\pm 1}(a) - a\| &< \varepsilon_n && (a \in P_n^{\varphi}), \\ |\varphi_{n+1}(b) - \psi_n(b)| &< \min\{\delta_n^{\psi}, \varepsilon_n\} && (b \in F). \end{aligned}$$

In the first line both signs are required. Restricting the second line to T_n^{ψ} gives

$$|\psi_n(t) - \varphi_{n+1}(t)| < \delta_n^{\psi} \qquad (t \in T_n^{\psi}),$$

which is **precisely the hypothesis for the next π_ψ application**. Restricting it instead to $R_n^{-1}(D_n)$ gives

$$\boxed{|\varphi_{n+1}(R_n^{-1}(d_j)) - \psi_n(R_n^{-1}(d_j))| < \varepsilon_n \quad (j \le n).}$$

This second restriction, not merely closeness on the matrix units, will give $\varphi(\alpha_n(d_j)) \to \psi(d_j)$.

5. **Construct r_{n+1} and verify the next induction hypothesis.** Since l_{n+1} is now fixed, obtain $T_{n+1}^{\varphi}, \delta_{n+1}^{\varphi}$ for $(\pi_\varphi, P_{n+1}^{\varphi}, \varepsilon_{n+1})$. Apply the π_ψ theorem enabled in the preceding step, using

$$\mu = \psi_n, \quad \nu = \varphi_{n+1}, \quad F = T_{n+1}^{\varphi}, \quad \gamma = \delta_{n+1}^{\varphi}.$$

For its output w_n, set

$$r_{n+1} = w_n r_n, \qquad R_{n+1} = R_n \circ \theta_{w_n}, \qquad \psi_{n+1} = \psi_n \circ \theta_{w_n}.$$

The conclusions are

$$\begin{aligned} \|\theta_{w_n}^{\pm 1}(a) - a\| &< \varepsilon_n && (a \in P_n^{\psi}), \\ |\psi_{n+1}(t) - \varphi_{n+1}(t)| &< \delta_{n+1}^{\varphi} && (t \in T_{n+1}^{\varphi}). \end{aligned}$$

The second line is the induction hypothesis at $n+1$. Thus every correction is supplied by the finite-set transport theorem, and every required stage-test hypothesis is the preceding correction's conclusion. The test sets are always chosen before the correction that must meet them.

6. **Derive all four Cauchy estimates.** For $j \le n$, protection of d_j and $L_n^{-1}(d_j)$ gives, respectively,

$$\begin{aligned} \|L_{n+1}(d_j) - L_n(d_j)\| &= \|v_n^* d_j v_n - d_j\| < \varepsilon_n, \\ \|L_{n+1}^{-1}(d_j) - L_n^{-1}(d_j)\| &= \|v_n L_n^{-1}(d_j) v_n^* - L_n^{-1}(d_j)\| < \varepsilon_n. \end{aligned}$$

The first equality uses that L_n is isometric. The second explains why $L_n^{-1}(D_n)$ was included in P_n^{φ}. The w_n estimates give the same two bounds for R_n and R_n^{-1}. Consequently, for any of the four sequences $Q_n = L_n, L_n^{-1}, R_n, R_n^{-1}$ and $q > p \ge j$,

$$\|Q_q(d_j) - Q_p(d_j)\| \le \sum_{k=p}^{q-1} 2^{-k} \le 2^{1-p}.$$

For arbitrary $a \in A$, isometry then gives

$$\|Q_q(a) - Q_p(a)\| \le 2\|a - d_j\| + 2^{1-p} \qquad (q > p \ge j).$$

First approximate a by a fixed d_j, then let $p \to \infty$. This proves the four pointwise Cauchy assertions. It does not assert norm convergence of the unitaries l_n, r_n.

7. **Form the inner automorphisms and check their actual inverses.** Define

$$\begin{aligned} u_n &= l_{n+1}^* r_n, \\ \alpha_n(a) &= u_n a u_n^* = L_{n+1}(R_n^{-1}(a)), \\ \alpha_n^{-1}(a) &= R_n(L_{n+1}^{-1}(a)). \end{aligned}$$

The previous step implies pointwise convergence of each of the four sequences, since A is complete. To see explicitly that the two compositions are Cauchy, use the following estimate. If F_n, G_n are pointwise Cauchy isometries, let $b = \lim_n G_n(a)$ and $c = \lim_n F_n(b)$. Then

$$\|F_n(G_n(a)) - c\| \le \|G_n(a) - b\| + \|F_n(b) - c\| \longrightarrow 0.$$

Apply this once with $(F_n, G_n) = (L_{n+1}, R_n^{-1})$ and once with $(F_n, G_n) = (R_n, L_{n+1}^{-1})$. It proves the required Cauchy conditions for $\alpha_n(a)$ and its **actual inverse** $\alpha_n^{-1}(a)$.

8. **Obtain the state convergence by substitution.** For $j \le n$, the boxed estimate in the construction of l_{n+1} becomes

$$\begin{aligned} |\varphi(\alpha_n(d_j)) - \psi(d_j)| &= |\varphi(L_{n+1}(R_n^{-1}(d_j))) - \psi(R_n(R_n^{-1}(d_j)))| \\ &= |\varphi_{n+1}(R_n^{-1}(d_j)) - \psi_n(R_n^{-1}(d_j))| \\ &< 2^{-n}. \end{aligned}$$

This is why the requested set contained $R_n^{-1}(D_n)$ and why the output pairs L_{n+1} with R_n, rather than with R_{n+1}. For every $a \in A$, contractivity of the states and isometry of α_n give

$$|\varphi(\alpha_n(a)) - \psi(a)| \le 2\|a - d_j\| + 2^{-n} \qquad (n \ge j).$$

Density, followed by $n \to \infty$, yields $\varphi(\alpha_n(a)) \to \psi(a)$. Together with innerness and the two Cauchy conditions, this is exactly $\mathrm{HasInnerIntertwiningSequence}(\varphi, \psi)$.

Main citations

- The stated existence or structural result · Exact source
- GNS vector-functional supplier with both representations irreducible · Exact source
- Main tool: state approximation with a protected finite set · Exact source
- Unrestricted finite-set approximation used to initialize the recursion · Exact source
- Ordered alternating corrections · Exact source
- Initialization without initial state closeness · Exact source
- Forward output is pointwise Cauchy · Exact source
- Actual inverse output is pointwise Cauchy · Exact source
- State control on dense tests · Exact source
- Extension of state convergence to all elements · Exact source
- Purity gives irreducibility of the GNS representation · Exact source
- The canonical GNS vector has norm one · Exact source

- The GNS vector functional is the original state · Exact source

Lean source signature (exact)

```
theorem hasInnerIntertwiningSequence_of_pure
    (phi psi : Limit →L[ℂ] ℂ)
    (hphi : MathlibAnnex.CStarAlgebra.IsPureState Limit phi) (hpsi : MathlibAnnex.CStarAlgebra.IsPureState
Limit psi) :
    HasInnerIntertwiningSequence phi psi
```

> Here `phi, psi` are φ, ψ. In the linked GNS proof, `rho, sigma` are π_φ, π_ψ, and `xi, eta` are their unit cyclic vectors. In the transition proof, `s.move` is the local theorem specialized using the displayed induction hypothesis. Its output `u` is v_n, while `hB` supplies the output `v` denoted w_n; `left' = u * s.left` and `right' = v * s.right` are l_{n+1} and r_{n+1}. The proof `hBclose` verifies the T_n^ψ inequality, and `hAclose` verifies the next T_{n+1}^φ inequality. These names occur in the linked proof, not in the displayed signature. `innerAt l` is $a \mapsto l^* a l$, and `outputUnitary n` is $l_{n+1}^* r_n$. The dense-state estimate is obtained by substituting $R_n^{-1}(d_j)$, exactly as displayed above.

Lean realization notes

The implementation also records paths for the local corrections, but this selected theorem only concludes the sequence property. Its proof does not presume a transporting endpoint automorphism or a general KOS principle. All choices below are made once within this construction.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:3dfe02286dcb40387b6decbe45dddfbd82a2a7dfe7a6612c592255e76b2fac33
Card revision: 1 · SHA-256: b7e0f73e3b318db4bc9c1cf34d55c3106c4a7e4dde0b330fddafafffc3c3c902
Exposition revision: 1 · SHA-256: 8a6ed21b4319688fca619a236dcd49f2620a74fd07e4ec36b855818b20475d35
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
Header SHA-256: 8d82ab0b3a186480c0fb043e7a0b4c6a465c7a41e37d5cf2a1b13c4e92f38c3b