A fixed representative family of CAR shell data

`MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily`

structure

One family records a transporting automorphism and exact shell links for each selected pure-state class, with an explicitly fixed root component.

Statement

Let A be the completed CAR algebra, the norm completion of the matrix stages $A_n = M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$, with the fixed coordinate reindexing. Write $\iota_n : A_n \to A$ for the canonical inclusion, $\omega(\iota_n(a)) = a_{00}$ for the root state, $f_n = \iota_n(E_{00})$ for its decreasing root projections, and $p_n = f_n - f_{n+1}$ for the difference shells. Each p_n is a self-adjoint projection. Let I be the set of unitary-equivalence classes of pure-state Gelfand–Naimark–Segal (GNS) representations of A. Choose a representative pure state φ_j for each $j \in I$, with the distinguished class $o = [\omega]$ represented by $\varphi_o = \omega$. A representative shell family assigns to each $j \in I$ an automorphism α_j and links $w_{j,n} \in A$ with $\varphi_j(\alpha_j(a)) = \omega(a)$ for every a, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$, and $w_{j,n} w_{j,n}^* = p_n$ for every n. At the distinguished class it further requires $\alpha_o = \mathrm{id}_A$ and $w_{o,n} = p_n$.

Definition

The component structure stores α_j, the equation $\varphi_j \circ \alpha_j = \omega$, the sequence $(w_{j,n})$, and its initial and final supports. The family structure chooses a component for every class and adds the root equations. They normalize a distinguished component of this chosen family; they do not impose equality between arbitrary choices made in unrelated existence statements.

Assumptions

The CAR system, root state, and selected representatives are fixed. A family is supplied as data of this structure. There is no restriction that I be countable, and this structure declaration does not itself prove the existence of such a family.

Conclusion

The fields are the component data for every class and the two root equations. Each component contains one automorphism, its state-transport equation, its link sequence, and both support equations. The root shell $p_n = f_n - f_{n+1}$ is distinguished from the nested flag projection f_n.

Main citations

- The stated existence or structural result · Exact source
- Exact component fields · Exact source
- Difference shell definition · Exact source
- Projection property of each difference shell · Exact source
- The identity component satisfies the required equations · Exact source

Lean source signature (exact)

```
structure RepresentativeShellFamily where
  data : ∀ j : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit, RepresentativeShellData j
  root_alpha : (data completedRootPureState.classOf).alpha =
    StarAlgEquiv.refl ℂ Limit
  root_link : ∀ n, (data completedRootPureState.classOf).link n = rootShell n
```

> Here `completedRootPureState` is the root state bundled with its purity proof. `data j` contains $(\alpha_j, (w_{j,n})_n)$ and the equations of `RepresentativeShellData j`. `root_alpha` and `root_link` fix the component at `completedRootPureState.classOf`, namely o. `rootShell n` is p_n, whereas `rootFlag n` is f_n. The separate exact excerpt below shows the component's full fields.

Related definition — separate exact excerpt

```
/-- Natural source data for one selected pure-state class.  Its fields are
only the automorphism and exact algebraic shell links obtained from KOS. -/
structure RepresentativeShellData (j : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit) where
  alpha : Limit ≃⋆ₐ[ℂ] Limit
  state_eq : ∀ a : Limit,
    (MathlibAnnex.CStarAlgebra.PureState.representative completedRootPureState j).1 (alpha a) =
      completedRootPureState.1 a
  link : ℕ → Limit
  initial_support : ∀ n,
    star (link n) * link n = alpha (rootShell n)
  final_support : ∀ n,
    link n * star (link n) = rootShell n
```

Lean realization notes

No rank-one, capture, simplicity, compactness, or approximate-innerness conclusion is stored in this structure. A later constructor can use approximate innerness to build the links, but that construction property is not an extra field required of every supplied family.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:3e800cc2482d039e0c10ab2af30dfa967a720a6eaf1fd7a550ad235a099cef66
Card revision: 1 · SHA-256: 1c532d64db83192c067a4e726afa6d4c4cd941d770a97dbf2687faf8544821f7
Exposition revision: 2 · SHA-256: ab4382383328c50b9b99ff1d004e2458b9c7fe0f4829510848d1999feb971733
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
Header SHA-256: d2bff7ed9ea98d6f53b55a4ac0b96a8f5507f8f1903e93f2cd488b7a91dbbad1