A geometric-structure homeomorphism transports the irreducible spectrum

`MathlibAnnex.Analysis.CStarAlgebra.irreducibleSpectrumHomeomorphOfGeometricStructure`

def

Transports the irreducible spectrum along a homeomorphism of geometric structure spaces.

Statement

Let $A, B \neq \{0\}$ be complex C*-algebras, possibly without identities. Suppose

$$e : (S(A), c_A) \longrightarrow (S(B), c_B)$$

is a homeomorphism of their geometric structure **closure spaces**. Both e and its inverse are continuous for the specified closure operators, with continuity tested on every subset.

The map e determines an ordinary homeomorphism

$$H_e : \widehat{A} \longrightarrow \widehat{B}$$

between the independent irreducible spectra with their kernel-induced ordinary topologies.

Definition

For $D = A, B$, $S(D)$ denotes kernels of extreme points of the closed unit ball of the Banach dual, with real convexity, and $c_D(\mathcal{U}) = \{I : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$. The extreme-functional kernel characterization gives its geometric identification. Let $\mathrm{GS}(D)$ be the component space and $\Phi_D : \widehat{D} \to \mathrm{GS}(D)$ the specified spectral homeomorphism.

The induced component homeomorphism is $G(e) : \mathrm{GS}(A) \to \mathrm{GS}(B)$, with $C \mapsto e(C)$ and inverse $D \mapsto e^{-1}(D)$. Full closure continuity gives ordinary continuity for the specified topologies.

The defining map and its inverse, with composition read from right to left, are

$$H_e = \Phi_B^{-1} \circ G(e) \circ \Phi_A, \qquad H_e^{-1} = \Phi_A^{-1} \circ G(e)^{-1} \circ \Phi_B.$$

Thus the three stages go from $\widehat{A}$ to $\mathrm{GS}(A)$, then $\mathrm{GS}(B)$, then $\widehat{B}$; the last stage uses the inverse of Φ_B.

Assumptions

The source requires nonzero complex C*-algebras A, B, each with its compatible order, and the specified full closure-space homeomorphism e. The underlying Hilbert spaces of irreducible representations are not fixed as additional inputs; the independent spectra use their pure-state

GNS presentation. The map e has domain $S(A)$, whose points are subsets of A; it is not a map $A \to B$ and no linear or multiplicative property on algebra elements is assumed.

Conclusion

The output is this specific ordinary homeomorphism between the independent irreducible spectra. It uses the given full closure-space homeomorphism and the two fixed spectral comparisons. The input e acts on geometric points, which are subsets of algebras, not on algebra elements or representation operators.

Main citations

- Exact full definition: `irreducibleSpectrumHomeomorphOfGeometricStructure`.
- The ordinary spectral comparison: the two endpoint homeomorphisms and their exact topologies.
- Normed/GeometricStructure/SpectrumMap.lean, `geometricSpectrumHomeomorph`, `geometricSpectrumHomeomorph_val`, `geometricSpectrumHomeomorph_toGeometricSpectrum`, and `image_geometricSpectrumClosure`: the induced component map and full closure transport.
- The concrete geometric-spectrum closure: the component-intersection closure used by this construction.
- Ryotaro Tanaka, arXiv:2504.05551v1, Corollary 5.13: the spectral consequence of a geometric-structure homeomorphism.
- Component-map construction and full closure continuity.

Lean source signature (exact)

The full definition below is copied from the pinned source.

```
def irreducibleSpectrumHomeomorphOfGeometricStructure : IrreducibleSpectrum A ≃ₜ IrreducibleSpectrum B :=
  (irreducibleSpectrumHomeomorph A).trans
    (((NormedSpace.geometricSpectrumHomeomorph e).toTopologicalHomeomorph
      (isTopologizable_geometricSpectrumClosure A) (isTopologizable_geometricSpectrumClosure B)).trans
        (irreducibleSpectrumHomeomorph B).symm)
```

In the source	Mathematical meaning
The two ambient algebras and independent universes **Exact declaration**	The independently specified nonzero complex C*-algebras A, B with compatible orders; no units are required.
Exact declaration `e` in the surrounding scope	The supplied full closure-space homeomorphism of geometric points, not a map on algebra elements.

In the source	Mathematical meaning
Exact declaration `(irreducibleSpectrumHomeomorph A).trans`	The first stage is $\Phi_A : \widehat{A} \to \mathrm{GS}(A)$.
Exact declaration `NormedSpace.geometricSpectrumHomeomorph e`	The middle stage is the component map $C \mapsto e(C)$ with its inverse.
Exact declaration `.toTopologicalHomeomorph`	The existing topologizability results give ordinary continuity of that same component map.
Exact declaration `(irreducibleSpectrumHomeomorph B).symm`	The final stage is $\Phi_B^{-1} : \mathrm{GS}(B) \to \widehat{B}$, giving the displayed composite H_e.

Exact surrounding source context (separate excerpts)

Source lines 27–27:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A] [Nontrivial A]
```

Source lines 28–28:

```
variable {B : Type v} [NonUnitalCStarAlgebra B] [PartialOrder B] [StarOrderedRing B] [Nontrivial B]
```

Source lines 29–30:

```
variable (e : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
  (NormedSpace.geometricStructureClosure ℂ B))
```