The intersection of a pure-state component is its GNS representation kernel

`MathlibAnnex.Analysis.CStarAlgebra.geometricSpectrumToPrimitive_toGeometricSpectrum_pureStateKernel_val`

theorem

Identifies a pure-state component intersection with the kernel of its GNS representation.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. A state on A is a positive bounded complex-linear functional of norm one. It is pure if it is an extreme point of the state space, regarded as a real convex set. Let φ be a pure state. Let $(H_\varphi, \widetilde{\pi}_\varphi, \xi_\varphi)$ be the GNS model of its canonical state extension $\widetilde{\varphi}(\lambda, a) = \lambda + \varphi(a)$ to the forced unitization $\mathbb{C} \oplus A$, and let $\pi_\varphi = \widetilde{\pi}_\varphi|_A$.

By the extreme-kernel theorem, $S(A)$ is the family of kernels $\ker \rho \subseteq A$ of bounded complex-linear functionals $\rho : A \to \mathbb{C}$ that are extreme points, for real convexity, of the entire closed dual unit ball B_{A^*}, with closure $c_A(\mathcal{T}) = \{I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I\}$. The pure-state kernel $\ker \varphi$ is one such point, by the pure-state extremality theorem. Let $C(\ker \varphi)$ denote its component for the finite-chain relation generated by equal points and nonclosed pairs. Write $\mathrm{GS}(A)$ for the set of these components. For a component C, put $J_C = \bigcap_{I \in C} I$, and write $\mathrm{PS}(A)$ for the family of all such intersections. The canonical map is

$$\gamma : \mathrm{GS}(A) \longrightarrow \mathrm{PS}(A), \qquad \gamma(C) = J_C.$$

Then, as subsets of the original algebra,

$$\gamma(C(\ker \varphi)) = J_{C(\ker \varphi)} = \ker \pi_\varphi. \qquad (1)$$

On the left we display the underlying subset of the primitive geometric point.

Assumptions

The given functional $\varphi \in A^*$ is positive, has norm one, and is extreme in the state space for real convexity; A carries its compatible C*-order. The GNS action is formed from this very state's canonical extension. The representation is fixed by φ; faithfulness is not assumed.

Conclusion

The whole component's intersection is the representation kernel $\ker \pi_\varphi$, using this fixed GNS representation. An individual functional kernel $\ker \varphi$ need not equal that intersection. Formula (1) is an equality of subsets; it gives neither injectivity of γ nor a comparison of topologies by itself.

Main citations

- Exact pure-state gamma identity: `geometricSpectrumToPrimitive_toGeometricSpectrum_pureStateKernel_val`.
- Component intersections, their family and the component-intersection map: the definitions of J_C, $\mathrm{PS}(A)$ and γ.
- Ryotaro Tanaka, arXiv:2504.05551v1, Lemma 5.10: the component-family and representation-kernel comparison.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem geometricSpectrumToPrimitive_toGeometricSpectrum_pureStateKernel_val
    (phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi) :
    (NormedSpace.geometricSpectrumToPrimitive ℂ A
      (NormedSpace.toGeometricSpectrum ℂ A
        (extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi
hpure⟩))).val =
      ((absoluteStateContinuousMap (NonUnitalStateGNS.representation phi hpure.1)).ker : Set A)
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A and its compatible order are fixed.

Given pure state `(phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi) :`	The given φ is a positive bounded complex-linear functional of norm one, extreme in the state space for real convexity. This same state determines the GNS model.

Kernel, component and its intersection
`(NormedSpace.geometricSpectrumToPrimitive ℂ A`
`  (NormedSpace.toGeometricSpectrum ℂ A`
`    (extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩))).val =`
	The innermost input is the geometric point $\ker\varphi$, with its extremality certificate. Sending it to its component gives $C(\ker\varphi) \in \mathrm{GS}(A)$. Applying γ and taking the underlying subset gives
	$$\gamma(C(\ker\varphi)) = J_{C(\ker\varphi)} \subseteq A.$$
	The nested applications act on this same point in that order.

The component map
`def toGeometricSpectrum (I : GeometricStructureSpace 𝕜 E) : GeometricSpectrum 𝕜 E :=`
`  (geometricStructureClosure 𝕜 E).toComponentPoint I`
	This map sends a point $I \in S(A)$ to its entire finite-chain component. In the preceding application, $I = \ker\varphi$.

In the source	Mathematical meaning

The intersection map
`def geometricSpectrumToPrimitive : GeometricSpectrum 𝕜 E → PrimitiveGeometricSpectrum 𝕜 E :=`
 	
`ClosureSpace.HullKernel.componentPointToComponentKernelPoint (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)`
	This is $\gamma : \mathrm{GS}(A) \to \mathrm{PS}(A)$, sending a component C to $J_C = \bigcap_{I \in C} I$. Its value is a subset of the algebra, rather than a subset of $S(A)$.

Kernel of the same GNS representation `((absoluteStateContinuousMap (NonUnitalStateGNS.representation phi hpure.1)).ker : Set A) := by`	The right side is $\ker \pi_\varphi \subseteq A$. The GNS action is formed from the canonical extension of this same state, and the continuous-linear-map wrapper changes no values. Thus (1) identifies the whole component intersection with the representation kernel, not with the individual functional kernel.
Operator-valued linear map `private def absoluteStateContinuousMap {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H] (pi : NonUnitalCStarRepresentation A H) : A →L[ℂ] (H →L[ℂ] H) := LinearMap.mkContinuous { toFun := pi, map_add' := map_add pi, map_smul' := map_smul pi }`	The wrapper has domain A and codomain $\mathcal{B}(H_\varphi)$. Its function is the given representation π_φ, with the same addition and complex-scalar action. It supplies the operator-valued map whose kernel appears on the right of (1).

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

The continuous-map wrapper in the same source is copied separately; its forward function is the given action; the continuous-map constructor receives the existing contractive bound:

```
private def absoluteStateContinuousMap {H : Type v}
    [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
    (pi : NonUnitalCStarRepresentation A H) : A →L[ℂ] (H →L[ℂ] H) :=
  LinearMap.mkContinuous
    { toFun := pi, map_add' := map_add pi, map_smul' := map_smul pi }
    1 (fun a => by
      change ‖pi a‖ ≤ 1 * ‖a‖
      simpa only [one_mul] using NonUnitalStarAlgHom.norm_apply_le pi a)
```

Related exact source for the reading guide

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

Kernel of the same GNS representation · Fixed source

```
      ((absoluteStateContinuousMap (NonUnitalStateGNS.representation phi hpure.1)).ker : Set A)
:= by
```

The component map · Fixed source

```
def toGeometricSpectrum (I : GeometricStructureSpace 𝕜 E) : GeometricSpectrum 𝕜 E :=
  (geometricStructureClosure 𝕜 E).toComponentPoint I
```

The intersection map · Fixed source

```
def geometricSpectrumToPrimitive : GeometricSpectrum 𝕜 E → PrimitiveGeometricSpectrum 𝕜 E :=
  ClosureSpace.HullKernel.componentPointToComponentKernelPoint
    (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)
```