The product-vector state on the completed CAR algebra

`MathlibAnnex.CStarAlgebra.CAR.rootState`

def

Compatible evaluation at the distinguished matrix coordinate extends continuously to the CAR completion.

Statement

Let A be the completed CAR algebra: the norm completion of the increasing union of $A_n = M_{2^n}(\mathbb{C})$ under the unital embeddings $a \mapsto a \otimes I_2$ (with the fixed coordinate reindexing). Write $\iota_n : A_n \to A$ for the canonical isometric inclusion and $e_{ij}^{(n)} = \iota_n(E_{ij})$ for its matrix units. At each stage take $a \mapsto a_{00}$, the vector functional of the distinguished coordinate vector. These functionals determine a continuous linear functional $\omega : A \to \mathbb{C}$ satisfying $\omega(\iota_n(a)) = a_{00}$ for every n and $a \in A_n$. The definition called the root state is this extension.

Definition

The coordinate estimate $|a_{00}| \leq \|a\|$ bounds each stage functional by 1. The standard inclusion preserves the zero coordinate, so these maps agree on common representatives in the algebraic direct limit. They therefore define a bounded linear functional on the normed direct limit. Extending that functional to its completion gives ω; the extension keeps the stage values unchanged. Positivity passes from the stage functionals to the completion by continuity, and evaluation at the common unit gives normalization.

Assumptions

The stage system and distinguished zero coordinate are fixed as above. The completion uses the CAR norm. No ambient Hilbert-space representation is chosen in this definition.

Conclusion

The defining restriction formula determines ω uniquely among continuous linear functionals, because the stage union is dense. The cited positivity and normalization results give $\omega(a^*a) \geq 0$ and $\omega(1) = 1$, so this functional is a state.

Main citations

- Definition and its exact construction · Exact source
- Stage coordinate functional · Exact source
- Compatibility of the coordinate functional · Exact source
- Bounded functional before completion · Exact source
- Stage restriction of the extended functional · Exact source
- Normalization of the root state · Exact source
- Positivity of the root state · Exact source
- Uniqueness from the dense stage restrictions · Exact source

Lean source signature (exact)

```
noncomputable def rootState : Limit →L[ℂ] ℂ :=
  preRootFunctional.extend (UniformSpace.Completion.toComplL : PreCAR →L[ℂ] Limit)
```

> Here `Limit` is A and `PreCAR` is the normed direct limit before completion. `preRootFunctional` is its bounded coordinate functional, and `.extend` gives ω on the completion. The stage inclusion ι_n used in the Statement is named `ofStage n` in the separately linked restriction theorem `rootState_stage`, not in the defining line displayed here. Positivity and normalization are also separate cited results.

Lean realization notes

Normalization and positivity are properties proved after the definition. The state evaluates one distinguished coordinate; it is not the normalized trace on a matrix stage. Its definition does not assert that it is faithful.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:40a210ab270847d874db7bbe7090d71a22bdf29970b3f0c379b59095ad3fb767
Card revision: 1 · SHA-256: 57b95199be75ec60e4eb22a021a39874e3414c50b4cdd2f9cf26aef0f56e9658
Exposition revision: 1 · SHA-256: 1cf9d495caf1f451824a59c31c133a84868a848b7374b520c4e7f8f1ea07fb49
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
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