Compact-operator model from a representative of the unique irreducible-representation class

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.faithful_and_compactOperatorModel_of_singleton`

Shows that a representative of the unique irreducible-representation class on a separable Hilbert space is faithful and has image exactly K(H).

Statement

Let A be a nonzero complex C*-algebra, not assumed unital, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. No separability of A, prior faithfulness, or simplicity is assumed. π is injective and satisfies the full unbundled compact-operator model: every π(a) is compact and every compact T on H has an exact preimage in A.

Assumptions

Let A be a nonzero complex C*-algebra, not assumed unital, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. No separability of A, prior faithfulness, or simplicity is assumed.

Conclusion

π is injective and satisfies the full unbundled compact-operator model: every π(a) is compact and every compact T on H has an exact preimage in A.

Proof idea

The source combines the faithfulness theorem for the unique-class representative with the compact-operator-model theorem. The latter gives compactness of every image and a preimage for every compact operator. This establishes the stated unbundled property; it does not construct a bundled StarAlgEquiv.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : NonUnitalCStarAlgebra A] [inst_1 : PartialOrder A] [StarOrderedRing A] {H : Type v} [inst_3 : NormedAddCommGroup H] [inst_4 : InnerProductSpace ℂ H] [inst_5 : CompleteSpace H] [Nontrivial A] [TopologicalSpace.SeparableSpace H] (pi : MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation A H), pi.IsSingletonIrreducibleModel → Function.Injective ⇑pi ∧ MathlibAnnex.Analysis.CStarAlgebra.IsCompactOperatorModel pi

2. Injective π ∧ IsCompactOperatorModel π, i.e. Injective π ∧ (∀ a, IsCompactOperator (π a)) ∧ (∀ T, IsCompactOperator T → ∃ a, π a=T).

3. The source combines the faithfulness theorem for the unique-class representative with the compact-operator-model theorem. The latter gives compactness of every image and a preimage for every compact operator. This establishes the stated unbundled property; it does not construct a bundled StarAlgEquiv.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.IsCompactOperatorModel

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_preimage_of_compact_singleton

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.isCompactOperator_map_of_singleton

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem faithful_and_compactOperatorModel_of_singleton [Nontrivial A]
[TopologicalSpace.SeparableSpace H]
(pi : NonUnitalCStarRepresentation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi) :
Function.Injective pi ∧ IsCompactOperatorModel pi
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: 4448c461ec5cd20bf403d08c42baa0de790b14987f7d2ab766036cdccc4b7537

Card SHA-256: f98a893acfc2db2d884a3079ff3eb5aa5de2a9cdd0e9a5e0ae65b26cb3ac687f

Approved exposition SHA-256: 4929773e9d8a058d13d40e87d8ae96b72bf6b36178ddf2baeb6620a8de91f2f5

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.