MATHLIBANNEX / CANONICAL DECLARATION CARD

Faithfulness of the target-state GNS representation

`MathlibAnnex.CStarAlgebra.CAR.tracialRepresentation_injective`

theorem

The ideal dichotomy of the same target forces the constructed nonzero representation to be injective.

Statement

Let A be the completed CAR algebra and τ its normalized trace. Fix a representative shell family and its chosen unitary links L_j in the selected atomic representation π. Write $T = C^*(\pi(A), \{L_j\})$ for the resulting concrete target and $\iota : A \to T$, $\iota(a) = \pi(a)$, for its faithful unital source map. This fixes one target and one source map throughout. Choose the state extension $\phi : T \to \mathbb{C}$ of τ supplied by state extension along ι. Its GNS representation is $(H_\phi, \rho_\phi, \xi_\phi)$, with

$$\|\xi_\phi\| = 1, \qquad \phi(t) = \langle \xi_\phi, \rho_\phi(t)\xi_\phi \rangle.$$

The inner product is linear in its second argument. The T-orbit of ξ_ϕ is dense by the GNS construction. The representation $\rho_\phi : T \to B(H_\phi)$ is injective.

Assumptions

The representative shell family is fixed. The established closed two-sided ideal dichotomy of this same T is available as a theorem. The GNS space is nonzero because its canonical vector has norm 1.

Conclusion

If $\rho_\phi(t) = \rho_\phi(s)$, then $t = s$. Thus the GNS representation of the chosen state on T is faithful as an algebra representation.

Proof route

Apply the closed-ideal dichotomy to the kernel and exclude the whole-algebra case using the unit on the nonzero GNS space.

Proof steps

1. The kernel $I = \ker \rho_\phi$ is a norm-closed two-sided ideal. The previously established result `shellFamilyTarget_closedIdeal_dichotomy` gives $I = \{0\}$ or $I = T$.

2. If $I = T$, then $\rho_\phi(1) = 0$. But ρ_ϕ is unital, so

   $$\xi_\phi = I_{H_\phi}\xi_\phi = \rho_\phi(1)\xi_\phi = 0,$$

   contradicting $\|\xi_\phi\| = 1$. Hence $I = \{0\}$, which is precisely injectivity. The source applies `Representation.injective_of_closed_ideal_dichotomy` after installing the separately proved nontriviality of H_ϕ.

Main citations

- The stated existence or structural result · Exact source
- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- The target-state GNS definition · Exact source
- Nontriviality witnessed by the unit vector · Exact source
- Closed-ideal dichotomy of the fixed target · Exact source
- The kernel argument for a nonzero unital representation · Exact source

Lean source signature (exact)

```
theorem tracialRepresentation_injective (family : RepresentativeShellFamily) :
    Function.Injective (tracialRepresentation family)
```

> `Function.Injective (tracialRepresentation family)` concerns the map from T to bounded operators on H_ϕ. In the linked proof, `nontrivial_tracialHilbertSpace` supplies a nonzero codomain, and `shellFamilyTarget_closedIdeal_dichotomy` is passed to the generic faithful-representation lemma. Neither is a new assumption added to the displayed signature.

Lean realization notes

This does not infer faithfulness of an arbitrary vector state from faithfulness of its representation. State faithfulness requires its own argument. No irreducibility of ρ_ϕ, purity of ϕ, or separability of T is assumed.

▼ **Exact Card identity**

Language: en
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