A common root vector realizes every selected state through the generators

`MathlibAnnex.CStarAlgebra.CAR.exists_targetDefectVectorFamily`

theorem

Builds a compatible family of unit vectors from a surviving residual space and identifies their CAR vector states.

Statement

For the supplied shell family and nonzero irreducible representation ρ described below, there exist a unit vector $\zeta \in K$ and unit vectors $\eta_j \in K$, one for each $j \in I$, with the following properties for every n and $a \in A$:

$$\sigma(f_n)\zeta = \zeta, \qquad \langle \zeta, \sigma(a)\zeta \rangle = \omega(a).$$

For every $j \in I$ they satisfy

$$\sigma(f_{j,n})\eta_j = \eta_j, \qquad V_j\eta_j = \zeta, \qquad \langle \eta_j, \sigma(a)\eta_j \rangle = \varphi_j(a).$$

Here $\sigma = \rho \circ \iota$, and inner products are linear in the second argument.

Assumptions

Let A be the completed CAR algebra, obtained by completing the matrix stages $M_{2^n}(\mathbb{C})$ with embeddings $a \mapsto a \otimes I_2$. Let f_n be the image of the first diagonal matrix unit. Thus $f_0 = 1$ and the f_n form a decreasing sequence of projections. The root state ω has value a_{00} on a matrix a in any stage.

Choose a pure state φ_j in each unitary-equivalence class $j \in I$ of pure-state GNS representations, with $\varphi_o = \omega$ at the root index o. Denote its complete complex GNS space, unital star representation and unit cyclic vector by (H_j, π_j, ξ_j). Set $\mathcal{H} = \bigoplus_{j \in I} H_j$ and $\pi = \bigoplus_{j \in I} \pi_j$. The index set need not be countable.

A supplied shell family consists of unital complex star automorphisms α_j with $\varphi_j \circ \alpha_j = \omega$ and elements $w_{j,n} \in A$. Put $f_{j,n} = \alpha_j(f_n)$. Their support identities are $w_{j,n}^* w_{j,n} = f_{j,n} - f_{j,n+1}$ and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$.

Let L_j be unitaries on $\mathcal{H}$ satisfying $L_j \pi(f_{j,n} - f_{j,n+1}) = \pi(w_{j,n})$ for every j, n. Let $T \subseteq B(\mathcal{H})$ be the norm-closed unital star algebra generated by $\pi(A)$ and all L_j. Write $\iota : A \to T$ for $\iota(a) = \pi(a)$ and $g_j \in T$ for the element represented by L_j. For a unital complex star representation $\rho : T \to B(K)$ on a complete complex Hilbert space, put $\sigma = \rho \circ \iota$ and $V_j = \rho(g_j)$. Each V_j is unitary because ρ preserves the unit, products and adjoints. Write $F_j = \bigcap_{n \geq 0} \operatorname{ran} \sigma(f_{j,n})$ for the common fixed space of the j-th transported flag. The representation ρ is nonzero and irreducible, in the sense that its only closed reducing subspaces are $\{0\}$ and K. No root normalization $L_o = I$ and no prescribed action of L_j on the atomic cyclic vectors is required.

Conclusion

All chosen vector states share one root vector: the family can be defined by $\eta_j = V_j^* \zeta$. These pointed state realizations supply the vector data for the cyclic-sum capture theorem. The present result does not assert that an individual η_j is cyclic for all of K, or that the limiting fixed spaces are one-dimensional.

Proof route

Choose a unit vector in one nonzero limiting fixed space and apply its represented generator to obtain ζ in the root fixed space. The CAR norm-compression estimates identify both vector states. For every other index, pull this same ζ back by the corresponding unitary. The fixed-space equivalence in shell reconstruction places the pulled-back vector in the correct fixed space, and compression again identifies its state.

Proof steps

1. The nonvanishing theorem gives i_0 and $0 \neq x \in F_{i_0}$. Normalize x to $\eta^{(0)} = x/\|x\|$ and put $\zeta = V_{i_0}\eta^{(0)}$. Unitarity preserves norm one. Reconstruction maps F_{i_0} onto the root common range, so all $\sigma(f_n)$ fix ζ.

2. For every $a \in A$, the source estimates are $\|f_n a f_n - \omega(a) f_n\| \to 0$ and $\|f_{j,n} a f_{j,n} - \varphi_j(a) f_{j,n}\| \to 0$. Apply the contractive representation σ and take a matrix coefficient at a unit vector fixed by the corresponding projections. Self-adjointness and fixedness make that coefficient equal to the constant difference between its vector state and the indicated source state. Norm convergence forces the difference to be zero. In particular this identifies the state of ζ as ω.

3. Define $\eta_j = V_j^*\zeta$ for each j. The same reconstruction equivalence, used in the reverse direction, gives $\eta_j \in F_j$. Unitarity gives $\|\eta_j\| = 1$ and $V_j\eta_j = \zeta$; membership in the ranges of projections gives their fixedness.

4. Apply the transported compression estimate to this unit fixed vector. It yields $\langle \eta_j, \sigma(a)\eta_j \rangle = \varphi_j(a)$ for every $a \in A$ and every j. The construction uses one root vector throughout rather than separately choosing unrelated vector-state realizations.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_nonzero_targetFixedSpace · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_targetDefectVectors_of_fixedSpace_ne_bot · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_targetDefectVectors · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_targetShellReconstruction · Exact source
- MathlibAnnex.CStarAlgebra.CAR.tendsto_representative_transported_compression · Exact source
- MathlibAnnex.CStarAlgebra.CAR.tendsto_transported_compressionError · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_surjective_selectedAtomicCyclicIsometry · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem exists_targetDefectVectorFamily
    (family : RepresentativeShellFamily)
    (L : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit →
      MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState)
    (hLunit : ∀ i, L i ∈ unitary
      (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState))
    (hLsource : ∀ i n, (L i).comp (selectedAtomicRepresentation
      (transportedFlag family i n - transportedFlag family i (n + 1))) =
        representedShellLink family i n)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K] (rho : Representation (AtomicTarget L) K)
    (hrho : rho.IsIrreducible) :
    ∃ (eta_o : K) (eta : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit → K),
      ‖eta_o‖ = 1 ∧
      (∀ n, (restrictedRepresentation L rho) (rootFlag n) eta_o = eta_o) ∧
      Representation.vectorFunctional (restrictedRepresentation L rho) eta_o =
        rootState ∧
      ∀ i,
        ‖eta i‖ = 1 ∧
        (∀ n, (restrictedRepresentation L rho)
          (transportedFlag family i n) (eta i) = eta i) ∧
        (Unitary.linearIsometryEquiv
          (representedGeneratorUnitary L hLunit rho i) : K ≃ₗᵢ[ℂ] K) (eta i) =
            eta_o ∧
        Representation.vectorFunctional (restrictedRepresentation L rho) (eta i) =
          (MathlibAnnex.CStarAlgebra.PureState.representative completedRootPureState i).1
```

> Here `Limit` is A, `completedRootPureState` is the root state ω bundled with its purity proof, and `SelectedAtomicHilbert completedRootPureState` is the atomic Hilbert space $\mathcal{H}$. `eta_o` is the common root vector ζ, while `eta i` is the vector η_j at the source index `i` corresponding to the mathematical index j; `eta_o` does not mean `eta` evaluated at the root index. `vectorFunctional sigma eta` sends a to $\langle \eta, \sigma(a)\eta \rangle$. `representative completedRootPureState i` is the selected state φ_j at that same index, and `rootState` is ω. The same `eta_o` occurs in the generator equation for every index.

Lean realization notes

The common root vector ζ and the member η_o indexed by the root class have the same root state, but their equality is not imposed here. The additional normalization $L_o = I$, used in that cyclic-sum capture theorem, would give $V_o = I$ and hence $\eta_o = \zeta$. No orthogonality or surjectivity conclusion is included in this declaration.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

Card UID: lfh:lfh-declaration-card:sha256:4888d88a4c7efeb37acd75ea32c5fe01cbce3d3cac678038c63b5954797902df

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Exposition revision: 1 · SHA-256: d1de79eac694b1605b00d082f1e28e79c934d815c630d7c8d4a7f5e0771e5342

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