

# A faithful separable model, but no separable irreducible model

MathlibAnnex.CStarAlgebra.CAR.atomicCounterexampleEndpoint\_and\_exists\_separable\_faithful\_representation\_and\_no\_separable\_irreducible\_representation

Combines the ordinary counterexample and the separable-representation conclusions for one fixed algebra.

## Statement

The fixed unital  $C^*$ -algebra  $A$  is simple and infinite-dimensional, has exactly one nonzero irreducible representation class and is not an algebra of compact operators. The same  $A$  has a faithful representation on the nonzero separable CAR trace-GNS space  $H\tau$ , but has no nonzero irreducible representation on any separable complex Hilbert space.

## Assumptions

Let  $A \subseteq B(H_{\text{at}})$  be the fixed unital  $C^*$ -algebra obtained by adjoining the chosen shell-link unitaries to the atomic representation of the CAR algebra  $C$ , and then taking the norm-closed unital  $*$ -algebra they generate. Let  $H\tau = L^2(C, \tau_C)$  be the GNS Hilbert space of the normalized CAR trace  $\tau_C$ . There are no additional theorem hypotheses. In particular, neither CH nor separable representability is assumed.

## Conclusion

The ordinary counterexample assertion holds for  $A$ . In addition,  $H\tau$  is nontrivial and separable, there exists an injective unital  $*$ -representation  $A \rightarrow B(H\tau)$ , and every possibly nonunital  $*$ -representation  $A \rightarrow B(K)$  on any separable complex Hilbert space  $K$  fails to be nonzero irreducible.

## Proof route

Combine the ordinary counterexample theorem, nontriviality and separability of the CAR trace-GNS space, the faithful tracial representation, and the universal exclusion of separable irreducible representations.

## Proof steps

1. Keep the entire ordinary assertion for the fixed  $A$ , including its arbitrary comparison universe.
2. Choose the tracial representation as the faithful separable witness.
3. Apply the universal separable irreducibility obstruction to every comparison map.

## Lean realization notes

The final existential witness and the universal comparison representations refer to the same  $A$ . The comparison Hilbert universe is independent. The conclusion excludes separable irreducible realizations, not all irreducible realizations; it does not assert that  $A$  is nonprimitive. The theorem has no tracial uniqueness clause: that is included separately in the counterexample theorem with its faithful separable tracial model.

Language: en · CARD\_CONTENT\_COMPLETE

## Main citations

[MathlibAnnex.CStarAlgebra.CAR.separableCounterexampleRepresentation\\_injective](#)

[MathlibAnnex.CStarAlgebra.CAR.not\\_isIrreducible\\_of\\_separable](#)

## Lean source declaration (exact)

```
-- The same fixed algebra has a faithful representation on a nontrivial separable
Hilbert space, but has no nonzero irreducible representation on any separable
Hilbert space in the arbitrary comparison universe. This does not assert that
the algebra is nonprimitive: the original irreducible model is retained. -/
theorem atomicCounterexampleEndpoint_and_exists_separable_faithful_representation_and_no_separa
ble_irreducible_representation :
  AtomicCounterexampleEndpoint.{v} ∧
  Nontrivial SeparableCounterexampleHilbertSpace ∧
  TopologicalSpace.SeparableSpace SeparableCounterexampleHilbertSpace ∧
  (∃ ρ : Representation AtomicCounterexampleAlgebra SeparableCounterexampleHilbertSpace,
  Function.Injective ρ) ∧
  (∀ (H : Type v) [NormedAddCommGroup H] [InnerProductSpace ℂ H]
  [CompleteSpace H] [TopologicalSpace.SeparableSpace H],
  ∀ ρ : NonUnitalRepresentation (A := AtomicCounterexampleAlgebra) (H := H),
  ¬ ρ.IsIrreducible) := by
  refine {shellFamilyEndpoint homogeneityShellFamily, nontrivial_traceHilbertSpace,
  separableSpace_separableCounterexampleHilbertSpace,
  (separableCounterexampleRepresentation,
  separableCounterexampleRepresentation_injective), ?_}
  intro H _ _ _ ρ
  exact not_isIrreducible_of_separable ρ
```

Read exact [MathlibAnnex v0.4.0 source](#)

## Exact Card identity

Stable Card ID: 496fb09fae36f8b07e1641152da97643e6c1b595cfc8bf458629debfabf9c241

Card SHA-256: 676839944ea44aba2ad0d82eecf10f9d7de497d78bdc437392982049cd4644ee

Approved exposition SHA-256: 33030ad83962e6456f2bf1888e972a45303ca1afba1c135ae85ea07d246dc816

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.