A separably represented C*-algebra with no separable irreducible representation

`MathlibAnnex.CStarAlgebra.CAR.atomicCounterexampleEndpoint_and_exists_separable_faithful_representation_and_no_separable_irreducible_representation`

theorem

The same algebra T has a faithful representation on a separable Hilbert space, but no nonzero irreducible representation on any separable Hilbert space.

Statement

Let A be the completed CAR algebra. Let $\pi : A \to B(\mathcal{H})$ be the direct sum of one chosen GNS representation from each pure-state GNS equivalence class, and let $(L_i)_{i\in I}$ be the unitaries obtained from the fixed CAR homogeneity and shell construction. Put $T = C^*(\pi(A) \cup \{L_i : i \in I\}) \subseteq B(\mathcal{H})$, and write $\iota(a) = \pi(a)$ for the embedding into T. Let τ be the normalized trace of A. The algebra T retains all the properties in the cited representation-classification theorem: it is nonzero, norm closed, unital, infinite dimensional and simple; ι is injective and unital; the inclusion $\rho_0 : T \to B(\mathcal{H})$ is faithful and irreducible; every nonzero irreducible $*$-representation of T is unitarily equivalent to ρ_0; and T is not isomorphic to the full algebra of compact operators on any Hilbert space. In addition, $H_\tau = L^2(A, \tau)$ is nonzero and separable, and there exists a faithful unital representation $T \to B(H_\tau)$. No separable complex Hilbert space carries a nonzero irreducible $*$-representation of T, even without assuming preservation of the identity.

Assumptions

All assertions concern the same generated algebra T and the same fixed shell unitaries. The absence of irreducible representations is quantified over arbitrary separable complex Hilbert spaces. Norm separability of T is neither required nor asserted.

Conclusion

T has both the faithful representation $\widetilde{\rho}$ on the separable space H_τ and the faithful irreducible representation ρ_0 on $\mathcal{H}$. The representation $\widetilde{\rho}$ is reducible, whereas $\mathcal{H}$ cannot be separable.

Proof route

Combine the classification and simplicity results for T with the normalized CAR trace vector, the dense CAR orbit, faithfulness of $\widetilde{\rho}$, and the theorem excluding nonzero irreducible representations on separable Hilbert spaces.

Proof steps

1. Let $(\pi_\tau, H_\tau, \xi_\tau)$ be the GNS representation and cyclic vector of τ. One has $\|\xi_\tau\| = 1$, so $H_\tau \neq \{0\}$. The continuous map $a \mapsto \pi_\tau(a)\xi_\tau$ has dense range. Since A is separable, H_τ is separable.
2. Take the GNS representation ρ_ϕ of the chosen extension ϕ of τ to T, and the unitary $E : H_\tau \to H_\phi$ from the cited trace construction. Then $\widetilde{\rho}(t) = E^{-1}\rho_\phi(t)E$ is the established faithful representation on H_τ. The cited theorem excluding nonzero irreducible representations on separable Hilbert spaces applies to

every $*$-representation $\rho : T \to B(H)$ with H separable. These statements are conjoined with the classification and structural properties of the same T.

Main citations

- The stated existence or structural result · Exact source
- The classification and structural properties of the constructed algebra · Exact source
- The fixed trace-space abbreviation · Exact source
- A nonzero vector in the trace space · Exact source
- Separability of that same space · Exact source
- The faithful witness · Exact source
- Exclusion on arbitrary separable Hilbert spaces · Exact source

Lean source signature (exact)

```
theorem
atomicCounterexampleEndpoint_and_exists_separable_faithful_representation_and_no_separable_irreducible_repre
sentation :
    AtomicCounterexampleEndpoint.{v} ∧
    Nontrivial SeparableCounterexampleHilbertSpace ∧
    TopologicalSpace.SeparableSpace SeparableCounterexampleHilbertSpace ∧
    (∃ ρ : Representation AtomicCounterexampleAlgebra SeparableCounterexampleHilbertSpace,
      Function.Injective ρ) ∧
    (∀ (H : Type v) [NormedAddCommGroup H] [InnerProductSpace ℂ H]
        [CompleteSpace H] [TopologicalSpace.SeparableSpace H],
      ∀ ρ : NonUnitalRepresentation (A := AtomicCounterexampleAlgebra) (H := H),
        ¬ ρ.IsIrreducible)
```

> The long conjunction in the exact signature keeps `AtomicCounterexampleEndpoint`, `Nontrivial` and `SeparableSpace` of H_τ, an existential injective representation, and the universal exclusion statement separate. `SeparableCounterexampleHilbertSpace` is H_τ and `NonUnitalRepresentation` does not initially require $\rho(1) = 1$. The proof chooses `separableCounterexampleRepresentation`, the same $\widetilde{\rho}$.

Lean realization notes

The faithful representation in the existential assertion is the already chosen $\widetilde{\rho}$. No different algebra or unrelated representation is chosen to establish the two separability assertions.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
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Card revision: 2 · SHA-256: 676839944ea44aba2ad0d82eecf10f9d7de497d78bdc437392982049cd4644ee
Exposition revision: 3 · SHA-256: 9f7c84764bbb25b92b7c4902e233aa74a31f37794608360c1a4c9f1be8bc4418
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