

CH and a Naimark counterexample of density $\aleph_1$

`MathlibAnnex.CStarAlgebra.CAR.continuum_eq_aleph_one_iff_existsNaimarkCounterexampleOfDensity`

Relates CH to the general density-existence predicate, not only to separably represented examples.

Statement

The continuum hypothesis holds if and only if there exists a nontrivial, possibly nonunital complex C^* -algebra of norm density $\aleph_1$ with one nonzero irreducible representation class and a displayed irreducible model that is not the full compact-operator algebra.

Assumptions

This is a biconditional with no extra hypothesis. CH means $\mathfrak{c} = \aleph_1$, where $\mathfrak{c} = 2^{\aleph_0}$. The existence predicate is the one described in the counterexample existence theorem, with density $\aleph_1$, and with carriers and comparison spaces in `Type`. No faithful separable representation is assumed for an arbitrary witness.

Conclusion

$\mathfrak{c} = \aleph_1 \Leftrightarrow$ there exists a Naimark counterexample of exact norm density $\aleph_1$ in the stated sense.

Proof route

Under CH, use the continuum-density example. Conversely, the general representation-theoretic density obstruction applied to a density- $\aleph_1$ witness gives $\mathfrak{c} \leq \aleph_1$; combine this with $\aleph_1 \leq \mathfrak{c}$.

Proof steps

1. Rewrite the density in the counterexample existence theorem using $\mathfrak{c} = \aleph_1$.
2. For the converse, use a dense set of cardinality $\aleph_1$ provided by the exact-density hypothesis.
3. The general nonunital density obstruction forces $\mathfrak{c} = \aleph_1$, without requiring a separable representation of the witness.

Lean realization notes

CH is a proposition in the biconditional, not an added axiom of the development. This is an object-level equivalence, not a forcing construction or a metatheoretic independence certificate. The supporting obstruction has broader universe parameters; the final biconditional here is its closed universe-zero form.

Language: en · CARD_CONTENT_COMPLETE

Main citations

[MathlibAnnex.CStarAlgebra.CAR.existsNaimarkCounterexampleOfDensity_continuum](#)

[MathlibAnnex.Analysis.CStarAlgebra.continuum_eq_aleph_one_of_existsNaimarkCounterexampleOfDensity](#)

Lean source declaration (exact)

```
/-- CH is equivalent to the existence of an ordinary possibly nonunital
Naimark counterexample of exact norm density aleph one. Carriers in this closed
existence statement lie in Type; the reverse obstruction is universe-polymorphic. -/
theorem continuum_eq_aleph_one_iff_existsNaimarkCounterexampleOfDensity :
  (Cardinal.continuum : Cardinal.{0}) = Cardinal.aleph 1 ↔
  ExistsNaimarkCounterexampleOfDensity (Cardinal.aleph 1 : Cardinal.{0}) := by
  constructor
  · intro hCH
  rw [← hCH]
exact existsNaimarkCounterexampleOfDensity_continuum
· exact continuum_eq_aleph_one_of_existsNaimarkCounterexampleOfDensity
```

Read exact [MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: 49b78f91c5e53fc7fff64fc366388f4d162fb65ecd6cb16fafe5785a8f2a236c

Card SHA-256: e9e119b142df5c73ca63aebd69db0c6601d2eba833dd69fa691bd0485a511bac

Approved exposition SHA-256: 5fa0d3f58c0ca6dff939a5b6fbd1f0f68257c58b99323a198beb0c90cab337d

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source-exposition correspondence, not an independent proof audit.