CH and a Naimark counterexample of norm density aleph one

`MathlibAnnex.CStarAlgebra.CAR.continuum_eq_aleph_one_iff_existsNaimarkCounterexampleOfDensity`

theorem

Characterizes the continuum hypothesis by the existence of a Naimark counterexample of exact norm density $\aleph_1$.

Statement

Write $\mathfrak{c} = 2^{\aleph_0}$. Then

$$\mathfrak{c} = \aleph_1 \iff \text{there exists a Naimark counterexample of norm density } \aleph_1.$$

The right-hand side has the following precise meaning. There are a nonzero, possibly nonunital complex C*-algebra B, a complex Hilbert space K, and a nonzero irreducible $*$-representation $\sigma : B \to B(K)$ such that every nonzero irreducible $*$-representation of B is unitarily equivalent to σ, the map σ is not both injective and surjective onto $\mathcal{K}(K)$, and $\mathrm{dens}_{\|\cdot\|}(B) = \aleph_1$. Preservation of an identity is not required of these representations.

Assumptions

The equivalence itself assumes neither CH nor its negation. In the reverse implication, B and σ are arbitrary objects satisfying the displayed existence statement; a faithful representation of B on a separable Hilbert space is not an additional hypothesis.

Conclusion

CH implies the stated existence, and any witness of that existence implies CH. The equivalence is a mathematical theorem about exact norm density; it neither proves CH nor supplies a forcing or metatheoretic independence proof.

Proof route

For the forward implication, substitute $\mathfrak{c} = \aleph_1$ in the continuum-density existence theorem. For the reverse implication, the general density obstruction for a C*-algebra with a single irreducible representation class gives $\mathfrak{c} \le \aleph_1$; combine it with $\aleph_1 \le \mathfrak{c}$.

Proof steps

1. Assume $\mathfrak{c} = \aleph_1$. The continuum-density existence theorem gives a nonzero C*-algebra with the stated representation properties and norm density $\mathfrak{c}$. Substitution yields density $\aleph_1$.

2. Conversely, let B, K, σ satisfy the right-hand side. Exact density provides a norm-dense $D \subseteq B$ with $|D| = \aleph_1$. Apply the cited density obstruction to σ, the unitary equivalence of all nonzero irreducible representations to it, its failure to be an injective representation onto $\mathcal{K}(K)$, and the density of D. This theorem allows a nonunital algebra and gives $\mathfrak{c} \le |D|$. No separable representation is required.

3. Therefore

$$\mathfrak{c} \leq |D| = \aleph_1 \leq \mathfrak{c}.$$

Antisymmetry gives $\mathfrak{c} = \aleph_1$, as required.

Main citations

- The stated existence or structural result · Exact source
- The continuum-density witness, without CH · Exact source
- The exact counterexample existence condition · Exact source
- The general reverse implication · Exact source
- The density obstruction for an arbitrary counterexample · Exact source

Lean source signature (exact)

```
theorem continuum_eq_aleph_one_iff_existsNaimarkCounterexampleOfDensity :
    (Cardinal.continuum : Cardinal.{0}) = Cardinal.aleph 1 ↔
      ExistsNaimarkCounterexampleOfDensity (Cardinal.aleph 1 : Cardinal.{0})
```

> `(Cardinal.continuum : Cardinal.{0})` is the continuum $\mathfrak{c}$ in the universe of cardinalities of `Type 0` types; `.{0}` specifies the universe, not the value zero. `Cardinal.aleph 1` is $\aleph_1$. The separately linked definition of `ExistsNaimarkCounterexampleOfDensity` supplies the exact existential quantifiers and the condition excluding an injective representation onto all compact operators. The proof first substitutes the CH equality in the continuum-density theorem, then invokes `continuum_eq_aleph_one_of_existsNaimarkCounterexampleOfDensity`. The general reverse theorem allows independent algebra and Hilbert-space universes; this statement uses its level-zero instance.

Lean realization notes

The forward implication uses the concrete algebra built from CAR. The reverse implication is not restricted to that construction.

▼ **Exact Card identity**

Language: en

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