

Every norm-dense subset of A has size at least the continuum

`MathlibAnnex.CStarAlgebra.CAR.continuum_le_cardinalMk_dense_atomicCounterexampleAlgebra`

Gives a lower bound for every dense subset, not merely for a selected generating set.

Statement

Every norm-dense subset D of the fixed CAR-based C^* -algebra A has cardinality at least $\mathfrak{c} = 2^{\aleph_0}$.

Assumptions

Let $A \subseteq B(H_{\text{at}})$ be the fixed unital C^* -algebra obtained by adjoining the chosen shell-link unitaries to the atomic representation of the CAR algebra C , and then taking the norm-closed unital $*$ -algebra they generate. Let $D \subseteq A$ and assume that D is dense in the norm topology of A .

Conclusion

$\mathfrak{c} \leq |D|$. In particular, A has no countable norm-dense subset.

Proof route

Apply the proved density obstruction for an infinite-dimensional algebra with one irreducible representation class to the atomic inclusion of A .

Proof steps

1. Use the singleton irreducible-model property of the atomic inclusion.
2. Use the proved infinite-dimensionality of A .
3. Apply the density obstruction to the arbitrary dense set D .

Lean realization notes

The exact statement quantifies over $s : \text{Set } A$ and takes $hs : \text{Dense } s$. The conclusion is $\text{Cardinal.continuum} \leq \#s$, with $\#s$ the cardinality of the subtype. This is a norm-topology statement, not a weak-operator or strong-operator density statement. Faithful representability on a separable Hilbert space does not assert norm separability of A .

Language: en · CARD_CONTENT_COMPLETE

Main citations

`MathlibAnnex.CStarAlgebra.CAR.not_finiteDimensional_shellFamilyTarget`

`MathlibAnnex.Analysis.CStarAlgebra.Representation.continuum_le_cardinalMk_dense_algebra_of_singleton_of_not_finiteDimensional`

Lean source declaration (exact)

```
/-- No norm-dense subset of the fixed algebra is smaller than the continuum. -/
theorem continuum_le_cardinalMk_dense_atomicCounterexampleAlgebra
(s : Set AtomicCounterexampleAlgebra) (hs : Dense s) :
Cardinal.continuum ≤ #s :=
Representation.continuum_le_cardinalMk_dense_algebra_of_singleton_of_not_finiteDimensional
(shellFamilyInclusion homogeneityShellFamily)
(isUniqueIrreducibleModel_shellFamilyInclusion.{0} homogeneityShellFamily).2
(not_finiteDimensional_shellFamilyTarget homogeneityShellFamily) s hs
```

[Read exact MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: 4e9f1fd1b5609f4b4591912d2f2f242e022b38ef13ae18ba179ad2c30a17835f

Card SHA-256: 1bb9f1304e4319112a787cd90aab522b6c259ccfdcf7571dce1d28cba6b569b5

Approved exposition SHA-256: 323e76dd6614ce9cc63b512c78dab02417917090f9b8f6840997bd2d9945480d

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source-exposition correspondence, not an independent proof audit.