Finite projective incidence determines a semilinear equivalence

`Projectivization.exists_semilinearEquiv_of_finite_span`

theorem

Realizes a finite-incidence-preserving projective bijection by a semilinear equivalence.

Statement

Let K, L be fields, let V be a K-vector space, and let W be an L-vector space. Write $[v] = Kv$ for $v \in V \setminus \{0\}$ and $[w] = Lw$ for $w \in W \setminus \{0\}$. These are the points of the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(W)$, respectively. For $D \subseteq \mathbb{P}(V)$, let $\langle D \rangle_{\mathrm{p}}$ be the set of lines in the algebraic linear span of representatives of D. In particular, $\langle \varnothing \rangle_{\mathrm{p}} = \varnothing$.

Suppose $e : \mathbb{P}(V) \to \mathbb{P}(W)$ is a bijection and

$$p \in \langle D \rangle_{\mathrm{p}} \iff e(p) \in \langle e(D) \rangle_{\mathrm{p}} \qquad (D \text{ finite},\ p \in \mathbb{P}(V)). \tag{1}$$

If the cardinal-valued algebraic dimension of V over K is at least three, there exist a field isomorphism $\sigma : K \to L$ and an additive bijection $U : V \to W$ such that

$$U(av) = \sigma(a)U(v), \qquad e([v]) = [U(v)] \quad (a \in K,\ v \in V \setminus \{0\}). \tag{2}$$

The scalar identity also holds at $v = 0$. Thus one semilinear equivalence realizes the prescribed projective bijection on every projective point.

Assumptions

Incidence is both preserved and reflected for every finite family, not merely for the chosen basis rays. The dimension assumption means that three linearly independent vectors exist; it does not assert finite dimensionality. The spaces need no norms, inner products, or topology, and the two fields need not coincide. No assumption on the characteristic is made.

Conclusion

This is a version of the fundamental theorem of projective geometry.

The same pair (σ, U) works for all nonzero representatives. The semilinear equivalence includes its σ^{-1}-semilinear inverse $U^{-1} : W \to V$. In particular,

$$e^{-1}([w]) = [U^{-1}(w)] \qquad (w \in W \setminus \{0\}). \tag{3}$$

The declaration asserts existence, not uniqueness of the choices. It imposes no involution, compatibility with an involution, or continuity condition on σ.

Main citations

The finite-incidence realization theorem gives one field isomorphism and one semilinear equivalence realizing every projective point. The semilinear equivalence construction includes the inverse over the inverse field isomorphism.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem exists_semilinearEquiv_of_finite_span
    (e : ℙ K V ≃ ℙ L W)
    (he : ∀ (T : Set (ℙ K V)), T.Finite → ∀ p,
      p ∈ Subspace.span T ↔ e p ∈ Subspace.span (e '' T))
    (hdim : 3 ≤ Module.rank K V) :
    ∃ σ : K ≃+* L, ∃ U : V ≃ₛₗ[σ.toRingHom] W,
      ∀ (v : V) (hv : v ≠ 0), e (mk K v hv) =
        mk L (U v) (by simpa only [map_zero] using U.injective.ne hv)
```

In the source	Mathematical meaning
Exact declaration `K`, `L`, `V`, `W` and their ambient instances	Two fields and their vector spaces; neither finite dimensionality nor an analytic structure is present.
Exact declaration `e : ℙ K V ≃ ℙ L W`	The prescribed projective bijection, not a lift constructed in advance.
Exact declaration `he`, with `T.Finite` and arbitrary `p`	Both directions of (1), using the algebraic projective span, for every finite $D \subseteq \mathbb{P}(V)$ and every $p \in \mathbb{P}(V)$, including $D = \varnothing$.
Exact declaration `hdim : 3 ≤ Module.rank K V`	The cardinal-valued dimension $\dim_K V$ is at least three; infinite-dimensional spaces are allowed.
Exact declaration `∃ σ : K ≃+* L, ∃ U : V ≃ₛₗ[σ.toRingHom] W`	One field isomorphism $\sigma : K \to L$ and one additive bijection $U : V \to W$ with $U(av) = \sigma(a)U(v)$. Its inverse is semilinear over σ^{-1}.
Exact declaration `∀ (v : V) (hv : v ≠ 0), e (mk K v hv) = ...`	The equality $e([v]) = [U(v)]$ for every $v \in V \setminus \{0\}$, with the same σ and U throughout. The vector $U(v)$ is nonzero, so the right-hand projective point is defined.

The following source context is separate from the declaration prefix. The later comment at lines 535–536 is not part of this declaration.

Exact surrounding source context (separate excerpts)

Source lines 38–41:

```
universe uK uL uV uW uI
variable {K : Type uK} {L : Type uL} {V : Type uV} {W : Type uW}
variable [Field K] [Field L]
variable [AddCommGroup V] [Module K V] [AddCommGroup W] [Module L W]
```

Source lines 44–47:

```
local instance (σ : K ≃+* L) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=
  RingHomInvPair.of_ringEquiv σ
local instance (σ : K ≃+* L) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=
  RingHomInvPair.of_ringEquiv_symm σ
```

Exact declaration for the source guide

```
theorem exists_semilinearEquiv_of_finite_span
    (e : ℙ K V ≃ ℙ L W)
    (he : ∀ (T : Set (ℙ K V)), T.Finite → ∀ p,
      p ∈ Subspace.span T ↔ e p ∈ Subspace.span (e '' T))
    (hdim : 3 ≤ Module.rank K V) :
    ∃ σ : K ≃+* L, ∃ U : V ≃ₛₗ[σ.toRingHom] W,
      ∀ (v : V) (hv : v ≠ 0), e (mk K v hv) =
        mk L (U v) (by simpa only [map_zero] using U.injective.ne hv) := by
```