A coefficient-kernel component equals the normal part

`MathlibAnnex.Analysis.CStarAlgebra.component_restrictedMatrixCoefficientKernel_eq_normalPart`

theorem

Identifies the normal part with one component of the ambient geometric structure space.

Statement

Let H be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a nonzero norm-closed complex $*$-subalgebra acting irreducibly on H. No identity is required. For $x, y \in H \setminus \{0\}$, define the bounded linear functional

$$f_{x,y} : A \to \mathbb{C}, \qquad f_{x,y}(a) = \langle y, a(x) \rangle,$$

where the inner product is conjugate-linear in the first variable. The geometric structure space $S(A)$ consists of the kernels of extreme points of the closed unit ball of the Banach dual of A, with convexity over $\mathbb{R}$. The coefficient-kernel membership theorem guarantees that every $\ker f_{x,y}$ above belongs to $S(A)$. For $D \subseteq S(A)$, the hull–kernel closure is

$$c_A(D) = \left\{ I \in S(A) : \bigcap_{J \in D} J \subseteq I \right\}.$$

Two points are linked if they are equal or their two-point set is not closed. For $I \in S(A)$, let $C(I)$ be its component under finite chains of links, including I itself. Then, for every $x, y \in H \setminus \{0\}$,

$$C(\ker f_{x,y}) = N(A) = \{\ker f_{z,w} : z, w \in H \setminus \{0\}\}. \quad (1)$$

The set $N(A)$ is called the normal part of A.

Assumptions

The algebra A has its inherited operator norm and adjoint. Here irreducibility means that the only closed A-invariant subspaces of H are $\{0\}$ and H; nonzero action is ensured by $A \neq \{0\}$. The vectors x, y need not be unit vectors.

Conclusion

Thus $N(A)$ is one component of the ambient space $S(A)$, independent of the nonzero pair x, y used to specify the starting kernel. This does not assert $N(A) = S(A)$.

Main citations

- The normal-part definition: the membership condition with a single nonzero pair.

- Whole-component theorem: `component_matrixCoefficientKernel_eq_family`, the result retained from the preceding component analysis.
- Exact concrete component theorem: the equality for the inclusion representation.
- arXiv:2504.05551v1, Lemma 6.3 and Definition 6.4: the coefficient component and the normal-part terminology.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem component_restrictedMatrixCoefficientKernel_eq_normalPart
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :
    (NormedSpace.geometricStructureClosure ℂ S).component
      (restrictedMatrixCoefficientKernel S hS x y hx hy) = normalPart S
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning
Hilbert space `variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`	The complete complex Hilbert space H on which A acts.
Closed operator algebra `variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))]`	The source name S denotes the algebra $A \subseteq \mathcal{B}(H)$. The closedness assumption concerns this subset of bounded operators.

Inherited C*-algebra structure
`private abbrev normalPartCStarAlgebra : NonUnitalCStarAlgebra S := NonUnitalStarSubalgebra.nonUnitalCStarAlgebra S (h_closed := inferInstance)`
`attribute [local instance] normalPartCStarAlgebra`
	The closed subalgebra has the inherited C*-algebra structure. The local instance makes that structure available when forming $S(A)$ and its closure; it does not impose a new hypothesis.

Irreducibility and coefficient vectors `(hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S)) (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :`	The inclusion action of A is nonzero and irreducible. The two vectors are $x, y \in H \setminus \{0\}$, defining $\ker f_{x,y}$ in (1).
Ambient component equality `(NormedSpace.geometricStructureClosure ℂ S).component (restrictedMatrixCoefficientKernel S hS x y hx hy) = normalPart S := by`	The complete starting point is the coefficient kernel for the same A, x, y. The component is taken in the ambient space $S(A)$, and $$C(\ker f_{x,y}) = N(A).$$ It is not a component formed after restricting to $N(A)$, and it does not assert $N(A) = S(A)$.

In the source	Mathematical meaning
Coefficient-kernel point `def restrictedMatrixCoefficientKernel (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S)) (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) : NormedSpace.GeometricStructureSpace ℂ S := ⟨{a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0}, setOf_inner_apply_eq_zero_mem_geometricStructureFamily S hS x y hx hy⟩`	The constructed point has underlying subset $\ker f_{x,y}$; the stated irreducibility and nonzero-vector conditions certify its membership in $S(A)$. The coefficient-kernel evaluation records that same subset.
Normal part `def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) : Set (NormedSpace.GeometricStructureSpace ℂ S) := {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧ {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}`	The set $N(A)$ consists of geometric points $I \in S(A)$ for which there exist $z, w \in H \setminus \{0\}$ with $I = \ker f_{z,w}$. Under the present hypotheses every such kernel is a geometric point, as used in (1).

The local instance below is a consequence of the closed-subalgebra hypothesis. The following Image section and its new variables are not part of this theorem.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Exact surrounding source:

```
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))]
```

Exact surrounding source:

```
private abbrev normalPartCStarAlgebra : NonUnitalCStarAlgebra S :=
  NonUnitalStarSubalgebra.nonUnitalCStarAlgebra S (h_closed := inferInstance)

attribute [local instance] normalPartCStarAlgebra
```

Related exact source for the reading guide

Normal part · Fixed source

```
def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) :
    Set (NormedSpace.GeometricStructureSpace ℂ S) :=
  {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧
    {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}
```

Coefficient-kernel point · Fixed source

```
def restrictedMatrixCoefficientKernel
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) : NormedSpace.GeometricStructureSpace ℂ S :=
  ⟨{a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0},
    setOf_inner_apply_eq_zero_mem_geometricStructureFamily S hS x y hx hy⟩
```

Ambient component equality · Fixed source

```
    (NormedSpace.geometricStructureClosure ℂ S).component
      (restrictedMatrixCoefficientKernel S hS x y hx hy) = normalPart S := by
```