Approximating another vector state along a unitary path

`MathlibAnnex.CStarAlgebra.CAR.exists_unitary_crossRepresentation_path_approx`

theorem

A fixed irreducible CAR representation can approximate any vector state on finitely many elements by moving a given unit vector.

Statement

Let A be the completed CAR algebra: the norm completion of the increasing union of $A_n = M_{2^n}(\mathbb{C})$ under the unital embeddings $a \mapsto a \otimes I_2$ (with the fixed coordinate reindexing). Write $\iota_n : A_n \to A$ for the canonical isometric inclusion and $e_{ij}^{(n)} = \iota_n(E_{ij})$ for its matrix units. A representation is a unital complex star homomorphism into the bounded operators on a complex Hilbert space. Irreducibility means nonzero action and no proper nonzero closed reducing subspace. We use the inner product conjugate linear in its first argument and write $\varphi_{\pi,v}(a) = \langle v, \pi(a)v \rangle$ for the vector functional. Let $\pi : A \to B(H)$ be irreducible with $H \neq 0$, let $\sigma : A \to B(K)$ be any representation, and choose unit vectors $\xi \in H$ and $\eta \in K$. For every finite $F \subset A$ and $\varepsilon > 0$, there exist a unitary $u \in U(A)$ and a norm-continuous path from 1 to u such that $|\varphi_{\pi,\pi(u)\xi}(a) - \varphi_{\sigma,\eta}(a)| < \varepsilon$ for all $a \in F$.

Assumptions

The spaces H and K are complete complex Hilbert spaces. Both representations are unital; only π must be irreducible. There are no initial stage-state tests and no protected finite set.

Conclusion

The approximating state is obtained from the specified starting vector ξ by a unitary in the identity path component of $U(A)$. The approximation is on the prescribed finite set at the endpoint.

Proof route

Keep the given vectors $\xi \in H$ and $\eta \in K$ fixed. Purification supplies a new unit vector $\zeta \in H$ that realizes the target state on a sufficiently large stage m. To move ξ to ζ, use the fixed-stage transport lemma at stage 0, where its tests are automatic. These are two different stages: m controls approximation, whereas stage 0 supplies transport without any near-centrality requirement.

Proof steps

1. Choose m so that each $a \in F$ has an approximant $b_a \in \iota_m(A_m)$ with $\|a - b_a\| < \varepsilon/3$. Apply finite-stage purification to the given π, σ, and η at stage m. Its hypotheses hold because π is nonzero and irreducible, both representations are unital, and $\|\eta\| = 1$. It gives $\zeta \in H$ with $\|\zeta\| = 1$ and $\varphi_{\pi,\zeta}(b) = \varphi_{\sigma,\eta}(b)$ for all $b \in \iota_m(A_m)$. This is a new vector; the prescribed starting vector ξ is not replaced.

2. Use the fixed-stage exact-transport lemma, not the protected-finite-set theorem. For a prescribed stage n and a positive commutator tolerance, it supplies a $\delta > 0$ before the pair of unit vectors is chosen. Its hypothesis compares their vector functionals on the matrix units of that stage. Set $n = 0$ and take the auxiliary commutator tolerance to be 1; its commutator conclusion will not be needed.

3. The inputs to this transport lemma are $\xi, \zeta \in H$ in the same representation π, not ξ and η from different spaces. Since $A_0 = \mathbb{C}$ and $e^{(0)}_{00} = 1$, unitality and the two unit-vector norms give $|\varphi_{\pi,\xi}(e^{(0)}_{00}) - \varphi_{\pi,\zeta}(e^{(0)}_{00})| = |1 - 1| = 0 < \delta$. This is the only matrix-unit test at stage 0. Thus the lemma supplies u and a path p from 1 to u with $\pi(u)\xi = \zeta$. No comparison between these states on the larger stage m, and no norm closeness of ξ and ζ, is required.

4. For each $a \in F$, the two states agree at b_a and are contractive, so $|\varphi_{\pi,\zeta}(a) - \varphi_{\sigma,\eta}(a)| \leq 2\|a - b_a\| < 2\varepsilon/3 < \varepsilon$. Substitute $\zeta = \pi(u)\xi$ to obtain the asserted endpoint estimate.

Main citations

- The stated existence or structural result · Exact source
- Finite-stage purification of the target state · Exact source
- Exact transport path, used at stage zero · Exact source
- Representation and nonzero irreducibility conventions · Exact source
- Equivalence of the two irreducibility interfaces on a nonzero Hilbert space · Exact source

Lean source signature (exact)

```
theorem exists_unitary_crossRepresentation_path_approx
    {H K : Type*}
    [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
    [Nontrivial H]
    [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
    (rho : Representation Limit H) (hrho : StarAlgHom.IsIrreducible rho)
    (sigma : Representation Limit K) (xi : H) (eta : K)
    (hxi : ‖xi‖ = 1) (heta : ‖eta‖ = 1)
    (F : Finset Limit) {epsilon : ℝ} (hepsilon : 0 < epsilon) :
    ∃ u : unitary Limit, ∃ p : Path 1 u, ∀ a ∈ F,
      ‖Representation.vectorFunctional rho (rho (u : Limit) xi) a -
        Representation.vectorFunctional sigma eta a‖ < epsilon
```

> Here `rho` is π, `sigma` is σ, and `xi, eta` are the given unit vectors. `Path 1 u` is a continuous unitary path; the final bound is the endpoint estimate on `F`. In the linked proof, `zeta` is the new finite-stage realization of the state of `eta`; `htrans xi zeta` applies the fixed-stage lemma inside `rho` at stage zero. The signature imposes no initial comparison of the states of `xi` and `eta`.

Lean realization notes

The conclusion supplies a path but makes no near-centrality claim and no uniform approximation claim along that path. It does not identify the two representations or give exact equality of their states on all elements.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

Card UID: lfh:lfh-declaration-card:sha256:5d348f9c51669f9b42100a13ab054c40413f2898dbdcdcc888e5c04c5e3efa16

Card revision: 1 · SHA-256: ca7b92d9fb47c907d06289cd2500a60d26d2ae92daa125feae35b8795c8b1b18

Exposition revision: 1 · SHA-256: 9e5ffe2c1b5b2c5dd3b1441d8e6c427441f373797a66b5afffe55da44ec792ad

Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73

Header SHA-256: ba201ca41f9aeb157f79785a5da063918f166ac5bc80c8f075d32dc52347f404