Root compression becomes scalar in norm

`MathlibAnnex.CStarAlgebra.CAR.tendsto_norm_compressionError`

theorem

Extends an exact rank-one compression identity on finite stages to a pointwise norm limit on the completed algebra.

Statement

Let A be the completed CAR algebra with stage maps $\iota_n : M_{2^n}(\mathbb{C}) \to A$ and root state $\omega(\iota_n(a)) = a_{00}$. Put $f_n = \iota_n(E_{00})$, where E_{00} is the projection onto the first coordinate. Then, for every fixed $x \in A$, $\|f_n x f_n - \omega(x) f_n\| \longrightarrow 0$ as $n \to \infty$.

Assumptions

The stages are linked by $a \mapsto a \otimes I_2$, and A is their norm completion. The functional ω is the continuous root state. Each f_n is a self-adjoint projection of norm at most one. The element x is arbitrary but fixed while n tends to infinity.

Conclusion

Writing $E_n(x) = f_n x f_n - \omega(x) f_n$, the theorem gives $\lim_{n \to \infty} \|E_n(x)\| = 0$ for every $x \in A$. The error $E_n(x)$ is an element of A; its norm is the real-valued quantity that converges.

Proof route

If $y = \iota_m(a)$ lies in a finite stage, rank-one compression gives $E_n(y) = 0$ whenever $n \geq m$. For arbitrary $z \in A$, the projection norm bound and boundedness of ω give $\|E_n(z)\| \leq (1 + \|\omega\|)\|z\|$, uniformly in n. Approximate the fixed x by such a y, and apply this estimate to $x - y$.

Proof steps

1. For $\varepsilon > 0$, choose $y = \iota_m(a)$ with $\|x - y\| < \varepsilon/(\|\omega\| + 2)$, using density of the stages.
2. For $n \geq m$, the exact finite-stage compression identity gives $E_n(y) = 0$.
3. Linearity gives $E_n(x) = E_n(x - y)$. Hence $\|E_n(x)\| \leq (1 + \|\omega\|)\|x - y\| < \varepsilon$.
4. This bound holds for every $n \geq m$, which is the required norm limit.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rootState · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rootState_stage · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rootFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isStarProjection_rootFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.compressionError · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rootFlag_mul_ofStage_mul · Exact source

- MathlibAnnex.CStarAlgebra.CAR.compressionError_sub · Exact source
- MathlibAnnex.CStarAlgebra.CAR.norm_compressionError_le · Exact source
- MathlibAnnex.CStarAlgebra.CAR.compressionError_ofStage · Exact source
- MathlibAnnex.CStarAlgebra.CAR.dense_stageRange · Exact source

Lean source signature (exact)

```
theorem tendsto_norm_compressionError (x : Limit) :
    Filter.Tendsto (fun n => ‖compressionError n x‖) Filter.atTop (nhds 0)
```

> Here `Limit` is A, `rootState` is ω, `rootFlag n` is f_n, and `compressionError n x` is $E_n(x)$. The filter expression says that this real-valued sequence of norms tends to 0 as $n \to \infty$.

Lean realization notes

This is pointwise convergence in x, not convergence in operator norm uniformly over the unit ball. It also does not assert norm convergence of the projections f_n. The displayed projections have rank one at their defining finite stage; no rank-one assertion in an arbitrary representation of A is used.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:5d44ee0aca9e46791d96b1f135b1092607d2b62670313827b2f813e5fe6e9be3
Card revision: 1 · SHA-256: c2ac9201baafd4343fd842fed3bb480153e0b78995eb4dce3491d6475fdb9b00
Exposition revision: 1 · SHA-256: 4119ae48aa3d3cc4f60c5e5adbe7f4530c6ff48f027809564ab08eb487d7cb30
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
Header SHA-256: 80678107836d77972acbfed5b188f0c57a3a9a6511151dbb16b7a02d2b4e5420