The dependent co-direct sum

`MathlibAnnex.Analysis.CStarAlgebra.C0DirectSum`

abbrev

Names the algebra of bounded families that vanish at infinity.

Statement

Let ι be any index set and let A_i be a complex C*-algebra for each $i \in \iota$, with no identity required. The dependent c_0-direct sum is the algebra of families $x = (x_i)$ in the bounded product satisfying the vanishing condition below.

Definition

The ambient bounded product consists of families $x_i \in A_i$ with $\sup_i \|x_i\| < \infty$, equipped with its supremum norm. Write

$$\bigoplus_{i \in \iota}^{c_0} A_i = \{x : \forall \varepsilon > 0,\ \{i : \varepsilon \le \|x_i\|\} \text{ is finite}\}$$

inside that bounded product. Operations are coordinatewise, and the norm is inherited. This subalgebra is norm closed, with the inherited C*-algebra structure.

Assumptions

Each fiber has its given complex C*-algebra structure. No identity, countability, separability, nonzero-fiber or nonempty-index assumption is imposed.

Conclusion

The abbreviation denotes this c_0 subalgebra, with the vanishing condition imposed for every positive threshold.

Main citations

- Exact complete definition.
- VanishesAtInfinity: the complete finiteness predicate.
- starSubalgebra: the carrier and algebra fields.
- isClosed_starSubalgebra: norm closedness.
- nonUnitalCStarAlgebra: the inherited structure.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
abbrev C0DirectSum {ι : Type uι} (A : ι → Type uA)
    [∀ i, NonUnitalCStarAlgebra (A i)] :=
  C0DirectSum.starSubalgebra A
```

In the source	Mathematical meaning
Exact declaration `{ι : Type uι} (A : ι → Type uA)`	An arbitrary index set ι and its family of algebras $(A_i)_{i\in\iota}$.
Exact declaration `[∀ i, NonUnitalCStarAlgebra (A i)]`	Each fiber carries the specified complex C*-structure.
Exact declaration `C0DirectSum.starSubalgebra A`	The subalgebra $\bigoplus_i^{c_0} A_i$ of bounded families: $\{i : \varepsilon \le \|x_i\|\}$ is finite for every $\varepsilon > 0$. Its operations are coordinatewise and its norm is the supremum norm.

Exact surrounding source context (separate excerpts)

Source lines 34–36:

```
namespace MathlibAnnex.Analysis.CStarAlgebra

universe uι uA uH
```